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Axiomatic Characterization of the Cut-Vertex Transit function of a Graph

Manoj Changat

Department of Futures Studies, University of Kerala, Trivandrum - 695 581, India

Ferdoos Hossein Nezhad

Department of Futures Studies, University of Kerala, Trivandrum - 695 581, India

Peter F. Stadler *

Bioinformatics Group, Department of Computer Science, Universität Leipzig, Härtelstrasse 16-18, D-04107 Leipzig, Germany

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Abstract

In this paper we study the cut-vertex transit function of a connected graph G and discuss its betweenness properties. We show that the cut-vertex transit function can be realized as the interval function of a block graph and derive an axiomatic characterization of cut-vertex transit functions. We then consider a natural generalization to hypergraphs and investigate its basic properties.

Keywords: Transit function; Convexity, Cut vertices, Block graphs

Math. Subj. Class.: 05C38, 05C65, 05C69, 05C75, 05C99

1 Introduction

A *transit function* R defined on a non-empty set V is a function $R : V \times V \to 2^V$ satisfying the three axioms

*PFS is also affiliated with the Interdisciplinary Center for Bioinformatics, the German Centre for Integrative Biodiversity Research (iDiv) Halle-Jena-Leipzig, the Competence Center for Scalable Data Services and Solutions Dresden-Leipzig, the Leipzig Research Center for Civilization Diseases, and the Centre for Biotechnology and Biomedicine at Leipzig University; the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany; the Institute for Theoretical Chemistry, University of Vienna, Vienna, Austria; the Center of noncoding RNA in Health and Technology (RTH) at the University of Copenhagen; and the Santa Fe Institute, Santa Fe, NM

E-mail addresses: mchangat@gmail.com (Manoj Changat), ferdows.h.n@gmail.com (Ferdoos Hossein Nezhad), studla@bioinf.uni-leipzig.de (Peter F. Stadler)

(t1) $x \in R(x,y)$ for all $x, y \in V$,

(t2) $R(x,y) = R(y,x)$ for all $x, y \in V$,

(t3) $R(x,x) = \{x\}$ for all $x \in V$.

Transit functions on discrete structures were introduced by M. Mulder [11] to generalize the concept of betweenness in an axiomatic way. Intuitively, $R(x,y)$ can be interpreted as an interval delimited by x and y. Transit functions captured attention in particular on discrete sets endowed with some additional structure, such as graphs, partially ordered sets, hypergraphs, etc. Several types of interval functions that can be defined in terms of paths were studied in some detail. Most of the literature concerns the shortest path transit function

$$I(u,v) := \{w \in V | w \text{ lies on a shortest } uv\text{-path}\} \tag{1.1}$$

on a connected graph G, see e.g. [10, 13, 14, 12]. As alternatives in particular the induced path [9, 15, 6, 5, 4] and the all-paths transit functions [3] have been considered.

P. Duchet [8] considered the following notion of betweeness for graph and hypergraphs:

$$C(u,v) := \{w \in V | w \text{ lies on every } uv\text{-path}\} \tag{1.2}$$

On graphs, $C(u,v) = \{u,v\}$ whenever u and v are located in the same block, and $C(u,v) = V$ if u and v are located in different connected components. For a connected graph G we therefore have the equivalent definition [11]

$$C(u,v) = \{u,v\} \cup \{w | w \text{ is a cut vertex between } u \text{ and } v\} \tag{1.3}$$

Hence C is called the *cut-vertex transit function* of G. A similar relation with cut-vertices can be found for hypergraphs, see sect. 4.

2 Cut-vertex transit function of a graph G

2.1 Terminology and Notation

Let $G = (V,E)$ be a finite, simple graph with vertex set V and edge-set E. Two graphs $G = (V,E)$ and $H = (W,F)$ are *isomorphic* if and only if there is a bijection f from V to W such that for adjacent vertices $u, v \in V$ the images $f(u), f(v)$ are adjacent vertices in W.

Given G and a vertex $v \in V$, we write $G - v$ for the the graph obtained by removing v and all its incident edges. We say that v is a cut vertex in a connected graph G if G has at least one edge and $G - v$ is disconnected. A graph is 2-connected if it contains no cut vertex. A block of G is a maximal 2-connected subgraph. A clique is a complete subgraph. A graph is a *block graph* if all its blocks are cliques. The *block closure* G^* of a connected graph G is the graph obtained from G by joining two vertices whenever they are in the same block of G. Thus G^* is the block graph.

Let R be a transit function on V. The underlying graph G_R of R has vertex set V and $uv \in E$ is an edge of G if and only if $R(u,v) = \{u,v\}$. Note that if R is a transit function on G, then G_R need not be isomorphic with G, see [11] for counterexamples.

The transit graph G_t of a transit function R on V is defined as the graph with vertex set V and $uv \in E$ is an edge of G_t if there is no $x \neq u, v$ such that $R(u,x) \cap R(x,v) = \{x\}$.

The following betweenness axioms were considered by Mulder in [11]:

(b1) $x \in R(u,v), x \neq v \Rightarrow v \notin R(u,x)$,

(b2) $x \in R(u,v) \Rightarrow R(x,v) \subseteq R(u,v)$,

(b3) if $x \in R(u,v)$ and $y \in R(u,x)$, then $x \in R(y,v)$ for all u,v,x,y.

(b4) if $x \in R(u,v)$, then $R(u,x) \cap R(x,v) = \{x\}$ for all u,v,x,

(m) $x,y \in R(u,v) \Rightarrow R(x,y) \subseteq R(u,v)$.

2.2 Underlying graph and transit graph of C

We now study the relationships between the underlying graph G_C, the graph G and transit graph G_t of the cut-vertex transit function C of G.

Proposition 2.1. *Let C be a cut-vertex transit function of a connected graph G. Then the underlying graph G_C, the block closure G^* of G and transit graph of C are isomorphic.*

Proof. Since $C(u,v) = \{u,v\}$ if and only if u,v are in the same block of G, i.e., if and only if $uv \in E(G^*)$. Thus G_C and G^* are isomorphic.

Two vertices u and v are adjacent in G_t is there is no $x \in V$ such that $C(u,x) \cap C(x,v) = \{x\}$. Since x is a cut-vertex if and only if $\{x\}$ is the intersection of two blocks, u and v must be in different blocks of G_t. Thus if u and v are in the same block of G, they are adjacent in G_t. Thus G_t and G^* are isomorphic. □

Proposition 2.2. *The cut-vertex transit function C of a connected graph G satisfies axioms* (b1)*,* (b2)*,* (b3)*,* (b4)*, and* (m)*.*

Proof. (b1) Let $x \in C(u,v)$. i.e. x is a cut vertex lying between every uv-path. Therefore v can not lie in any ux-path in G. Hence v is not a cut-vertex separating u and x and so $v \notin C(u,x)$. Hence C satisfies axiom (b1).

(b2) Let $x \in C(u,v)$ and $y \in C(u,v)$. We aim to prove that $y \in C(u,x)$. Since $y \in C(u,v)$, y is a cut-vertex separating u and x in G. i.e., y lies within every ux-path in G. Since $x \in C(u,v)$, x lies on any uv-path in G. Since G is connected we have y also lies on every uv-path in G. Therefore y is a cut-vertex separating u and v in G. i.e., $y \in C(u,v)$. Hence C satisfies (b2) on G.

(b3) Let $x \in C(u,v)$ and $y \in C(u,x)$, for vertices $x \neq u, x \neq v, y \neq x$ and $y \neq u$. Then x is a cut vertex separating u and v and y is a cut vertex separating u and x. That is, y lies between every ux-path and x lies between every uv-path in G. Since G is connected, x lies between every yv-path in G. Hence the cut-vertex x separates vertices y and v in G. That is, $x \in C(y,v)$. Thus C satisfies axiom (b3).

(b4) Let $x \in C(u,v)$. If $u = v$, then by definition it follows $C(u,u) \cap C(u,v) = \{u\}$ and $C(u,v) \cap C(v,v) = \{v\}$. Therefore, assume $x \neq u, x \neq v$. That is, x is a cut vertex separating u and v. It is clear that if y is any cut vertex separating u and x, then y cannot be a cut vertex separating x and v. That is, $y \in C(u,x)$ implies that $y \notin C(x,v)$. Similarly, $y \in C(x,v)$ implies that $y \notin C(u,x)$. That is, $C(u,x) \cap C(x,v) = \{x\}$ and hence C satisfies (b4).

(m) Let x,y,u,v and w be five distinct vertices such that $x,y \in C(u,v)$ and $w \in C(x,y)$. That is, w is a cut-vertex separating x and y, x and y are cut- vertices separating u and v. That is, w lies between every xy-path and both x and y lie between every uv-path in G. Since G is connected, the cut-vertex w also lies between every uv-path in G and hence separates u and v. that is, $w \in C(u,v)$. □

Lemma 2.3. *[5] if R is a transit function that satisfies axioms* (b1) *and* (b2)*, then the underlying graph G_R of R is connected.*

Corollary 2.4. *If G is connected, then the underlying graph G_C of the cut-vertex transit function C of G is connected.*

Proof. Since C satisfies (b1) and (b2), Lemma 2.3 implies that G_C is connected. □

For any arbitrary transit function R, we define

$$R(u, v, w) := R(u, v) \cap R(v, w) \cap R(w, u) \tag{2.1}$$

The cardinality $|R(u, v, w)|$ for several of path-based transit function characterizes interesting graph classes. For instance, in terms of the shortest path function, the graph for which $|I(u, v, w)| > 0$ are the modular graph [12]. For the induced path function, $|J(u, v, w)| > 0$ determines the triangle-free graphs[2], and $|J(u, v, w)| = 1$ identifies the svelte graphs [9]. For the all-path functions, $|A(u, v, w)| > 0$ characterizes the connected graphs and $|A(u, v, w)| = 1$ determines the trees.

The following was observed in [11] without a proof:

Proposition 2.5. *([11]) Let G be a connected graph. Then for any three vertices of G holds $|C(u, v, w)| \leq 1$. G is a tree if and only if $|C(u, v, w)| = 1$.*

Proof. First assume that at least two of u, v, w, say u and v, lie in the same block B. Thus $C(u, v, w) \subseteq \{u, v\}$. At most one of them can be a cut vertex between B and the block containing w, i.e., $C(u, v, w)$ is either empty or a singleton. Now suppose u, v, w are distributed across three different blocks. First, one of the three, say v, might be in a block that is in between u and w. Now $C(u, v) \cap C(v, w) = \{v\}$, and we are done. Otherwise, there is either a unique cut-vertex x or a unique block B in between any pair of the three. In the latter case it is easy to see that the intersection $C(u, v, w)$ is empty.

Note that $C(u, v, w) = \emptyset$ whenever all u, v, w are located in the same block. Hence $|C(u, v, w)| = 1$ implies that all blocks are edges, i.e., G is a tree. Conversely, any three distinct vertices in a tree have a unique median, which is also a cut vertex. □

Lemma 2.6. *Let R be a transit function satisfying axioms* (b2) *and $|R(u, v, w)| \leq 1$, then R satisfies axiom* (b1) *and* (b4) *on V, and G_R is connected.*

Proof. Let $x \in R(u, v)$, suppose R does not satisfy axiom (b4). Then there exists at least one $y \neq x$, such that $y \in R(u, x) \cap R(x, v)$. Since $x \in R(u, v)$, by (b2), $R(u, x) \subseteq R(u, v)$ and $R(x, v) \subseteq R(u, v)$. Therefore $y \in R(u, v)$. Hence $|R(u, x, v)| > 1$, which violates $|R(u, v, w)| \leq 1$. Therefore R satisfies axiom (b4). Since R satisfies (t1), (t2), and (b4), R also satisfies (b1). Since R satisfies (b1) and (b2), G_R is connected by Lemma 2.3. □

3 Axiomatic characterization

We start from the following simple observation

Proposition 3.1. *If G is a block graph, then the shortest path transit function and the cut vertex transit function coincide.*

Proof. Let G be a block graph. If u, v are in the same block, then $I(u, v) = \{u, v\}$ since every block is a clique. Thus $I(u, v) = C(u, v)$. If u and v are in distinct blocks, then there is a unique sequence of blocks between them, which pair-wisely intersect in cut-vertices. Since two consecutive cut vertices are contained in the same block, they are adjacent in G, and thus the sequence of cut vertices form the unique shortest path connecting u and v in G. Therefore $I(u, v) = C(u, v)$. □

It is shown in [1] that R is the interval function of a block graph G, i.e., $R = I_{G_R}$ if and only if it satisfied (t1), (t2), (b1), (b2), and the additional axiom

(U*) $R(u, x) \cap R(x, v) = \{x\}$ implies $R(u, v) \subseteq R(u, x) \cup R(x, v)$, for all $u, v, x \in V$.

This implies

Theorem 3.2. *Let $R : V \times V \to 2^V$ be a function on V. Then R is the cut-vertex transit function of a graph G if and only if it satisfies* (t1)*,* (t2)*,* (b1)*,* (b2)*, and* (U*).

Proof. Suppose R satisfies (t1), (t2), (b1), (b2), and (U*). The characterization in [1] implies that R is the interval function of the block graph G_R. Proposition 3.1 now implies implies $C = I_{G_R}$, and since G_R is block graph, G_C, G_R, and $G_{I_{G_R}}$ are isomorphic. Conversely, suppose R is the cut-vertex transit function of a graph G. Then R is also the interval function of its block closure G^*. Being the interval function of a graph G implies (t1), (t2), (b1), and (b2). Since G^* is block graph, [1] implies that R also satisfies (U*). □

An alternative characterization can be obtained using $R(u, v, w)$.

Theorem 3.3. *Let $R : V \times V \to 2^V$ be a function on V. Then R is the cut-vertex transit function of a graph G if and only if it satisfies* (t1)*,* (t2)*,* (b2)*,* (U*)*, and $|R(u, v, w)| \leq 1$ for all $u, v, w \in V$.*

Proof. Suppose R satisfies (t1), (t2), (b2), (U*) and $|R(u, v, w)| \leq 1$ for all $u, v, w \in V$. By Lemma 2.6, R also satisfies (b1), and hence Theorem 3.2 implies that R is the cut-vertex transit function of a block graph G_R. Conversely, let R by the cut-vertex transit function of a graph. Therefore, $|R(u, v, w)| \leq 1$ for any $u, v, w \in V$ by Proposition 2.5. On the other hand, R is the interval function of a block graph by Proposition 3.1, and hence satisfies in particular (t1), (t2), (b2), and (U*). □

We conclude this section with the following remark which can be deduced from the results of this section.

Remark 3.4. The cut-vertex transit function C of a connected graph G corresponds to the interval function of its block closure G^*.

4 Cut Vertex Transit Function in Hypergraphs

4.1 Notation and Terminology

A hypergraph H consists of a set V of vertices and a set $E \subseteq 2^V$ of non-empty edges. A path in a hypergraph H is an alternating sequence of hyperedges $x_1e_1x_2e_2x_3 \ldots x_{k-1}e_{k-1}x_ke_kx_{k+1}$ such that $x_1 \in e_1$, $x_i \in e_{i-1} \cap e_i$, and $x_{k+1} \in e_k$ and for $i \neq j$ we have $x_i \neq x_j$ and $e_i \neq e_j$. Every edge e_i in a path thus contains at least two vertices. A hypergraph is connected if every pair of vertices is connected by a path. A path in H is called simple if $e_i \cap e_j = \emptyset$ for

$j \neq i, i \pm 1$. A cycle is a simple path, say $x_1e_1x_2e_2x_3 \dots x_{k-1}e_{k-1}x_ke_kx_{k+1}$, in which $x_1 = x_{k+1}$.

We will need the following simple observation.

Lemma 4.1. *Let P by a uv-path in H. Then there is a simple uv-path composed of a subset of the hyperedges of P.*

Proof. If P is simple, there is nothing to show. Otherwise, there is an edge e_j intersecting some edge e_i with $i < j - 1$ and $x_i' \in e_i \cap e_j$. Then $P' = ue_1x_1 \dots e_ix'e_jx_{j+1} \dots e_kv$ is again a uv-path in H whose edge set is a proper subset of P. Repeating this construction eventually yields a simple uv-path. □

In order to determine $C(u, v)$, therefore, it suffices to consider only simple uv-path. Now let P by a simple uv-path and consider a vertex $x \in P$. We note that x is either contained in a single edge, say e_j or in the intersection of two consecutive edges $e_i \cap e_{i+1}$. In either case, P can be subdivided into a ux-path P_1 and a xv-path P_2. In the first case, P_1 and P_2 share e_j, while in the second case they have no edge in common. Both P_1 and P_2 are again simple.

Paths can also be concatenated, provided they do not contain the same edge. Let P_1 and P_2 be a ux-path and an xv-path, respectively. Then their concatenation $P_1P_2 = ue_1 \dots e_jxe_1' \dots e_k'v$ is again a path provided no edges appear twice as we traverse from u to v. We also define a concatenation in which the last edge of P_1 and the first edge of P_2 coincides: For $P_1 = ue_1 \dots x_je^*x$ and $P_1 = xe^*x_1'e_2' \dots e_k'v$ we set $P_1 \bullet P_2 := ue_1 \dots x_je^*x_1'e_2' \dots e_k'v$. Note that although x does not appear explicitly in $P_1 \bullet P_2$, it is still contained in e^*.

The *strong vertex deletion* removes with a vertex y also all edges from H that contain y. As in the graph case we write $H - y$ for the resulting hypergraph. A *strong cut vertex* is a vertex whose strong deletion renders H disconnected [7]. That is, x is a strong cut vertex in H if and only if there are two distinct vertices $u \neq x$ and $v \neq x$ in H such that every uv-path contains an edge containing x. In this case, we say that x separates u und v.

As in the case of graphs, we consider the transit function so that for $u \neq v$ we have $x \in C(u, v)$ if every uv-path in the hypergraph contains an edge that contains x, i.e., if x is a strong cut vertex in H separating u and v. Since the definition of uv-paths is symmetric and u and v are contained in every uv-path, it is clear that C satisfies (t1) and (t2). By convention we set $C(x, x) = \{x\}$ for all x, i.e., C is a well-defined transit function.

Similar to the case of graphs, the interval function $I(u, v)$ of a hypergraph H is defined as the function which returns, for every pair of vertices u, v of H the set of all vertices lying on shortest uv-paths in H. Since If x lies on every path from u to v, then x also lies on every shortest uv-path. Therefore we have the following immediate remark.

Remark 4.2. The cut vertex transit function $C(u, v) \subset I(u, v)$. Unlike the graph case (see, Remark 3.4), the function C of H need not always coincide with the interval function I of some hypergraph. In fact, $C(u, v)$ can be a proper subset of $I(u, v)$. For example, if P_1 and P_2 are edge disjoint uv-paths (that is, no edge in P_1 is contained in P_2 and no edge in P_2 is contained in P_1) and P_1 a shortest uv-path containing the vertex x, then there is no strong cut-vertex separating u and v so that $C(u, v) = \{u, v\}$ and hence $C(u, v) \subsetneq I(u, v)$.

We have also the following remark.

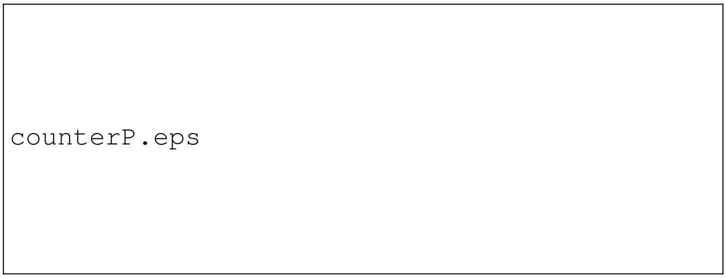

Figure 1: The absence of strong cut vertices does not imply the existence of two vertex disjoint paths. Every pair of xy-paths shares at least two vertices in the intersection of two of hyperedges with rank 4. Removal of all edges incident to any given vertex, however, still leave an xy-path behind, i.e., there is no strong cut vertex separating x and y.

Remark 4.3. As in graphs, the existence of two edge-disjoint paths is necessary – but not sufficient – to exclude strong cut vertices. We say that two xy-path P' and P'' are vertex-disjoint if their vertex sets only share the endpoints, i.e., $\bigcup_{e\in P'} \cap \bigcup_{e\in P''} = \{x, y\}$. In contrast to graphs, the existence of two vertex disjoint paths is necessary but not sufficient in hypergraphs to rule out strong cut vertices. In the example of Fig. 1, any two xy-paths share (at least) a pair of vertices located in the hyperedges of rank 4, i.e., there is no pair of vertex-disjoint xy-paths. Nevertheless, no vertex is contained in all xy-paths, and hence there is no strong cut vertex separating x and y. Therefore we have $C(x, y) = \{x, y\}$, and hence $C(x, y) \subsetneq I(x, y)$, in this example. It remains an interesting open problem to characterize the hypergraphs for which $C(x, y) = \{x, y\}$ for all $x \neq y$.

Properties of C on Hypergraphs

We start with a simple observation

Lemma 4.4. *Let x be a strong cut vertex separating u and v and let y by a strong cut vertex separating u and x. Then y also separates u and v. In other words, C satisfies the axiom* (b2).

Proof. By assumption, every uv-path contains an edge containing x, and thus a ux-subpath. Since every ux-path contains an edge containing y, this is also true for every uv-path. That is, In other words, C satisfies the axiom (b2). □

Lemma 4.5. *Let x be a strong cut vertex in H separating u and v. Let P_1 and P_2 be simple ux- and xv-paths, respectively. Then there are simple uv-paths P' and P'' that contain the edges of P_1 and P_2, respectively.*

Proof. First, we observe that the concatenations P_1P_2 or $P_1 \bullet P_2$ (in the case of equal end edges) are again paths in this case since all edges of P_1 except for the last one, e_1^*, are contained in the component of $H - x$ that contains u, all edges of P_2 except for the first one, e_2^* are contained in the component of $H - x$ that contains v. In particular, therefore $e_i \cap e_j = \emptyset$ for all edges $e_i \neq e_1^*$ in P_1 and $e_j \neq e_2^*$ in P_2. Thus, if $e_1^* = e_2^*$, then the concatenation $P_1 \bullet P_2$ is again simple, and the assertion follows. If $e_1^* \neq e_2^*$, then we have to consider the following case: (i) If P_1P_2 is simple, then the assertion follows trivially. (ii) otherwise, there is a minimal i such that there is a $x' \in e_i \cap e_2^*$ (in which case $ux_1 \dots x_i e_i x' P_2$ is a simple uv-path), or there is a maximal j such that $x'' \in e_2^* \cap e'_j$ (in which case $P_1 x'' e'_j \dots e'_k v$ is a simple xv-path). □

A useful consequence of this rather technical observation is:

Lemma 4.6. *Let x and y be two distinct strong cut vertices separating u and v. Then y is a strong cut vertex separating u and x or x and v.*

Proof. By assumption every (simple) uv-path contains an edge containing x and an edge containing y. By Lemma 4.5 the concatenation of any simple ux- and simple xv-path always contains y. Now suppose there are two simple uv-paths P_1 and P_2 such that y appears only in edges of the ux-subpath of P_1 and only in edges of the xv-subpath of P_2. Since P_1 and P_2 are simple, we can concatenate the xv-subpath of P_1 and the ux-subpath of P_2 to obtain a simple path that contains no edge containing u, a contradiction. Hence y is contained in every ux-path or in every xv-path. □

Lemma 4.6 can be translated into the following

Corollary 4.7. *Let C be the cut vertex transit function of a hypergraph. Then C satisfies axiom*

(U) *If $x \in C(u,v)$ implies $C(u,v) = C(u,x) \cup C(x,v)$.*

Lemma 4.8. *Let x and y be two distinct strong cut vertices in H such that x separates u and v and y separates u and x. Then y separates x and v.*

Proof. We first observe that both x and y are contained in every uv-path. We again invoke Lemma 4.5 to argue that every simple uv-path can be subdivided into a ux- and xv-path, with the property that either every ux-path contains an edge containing y. Furthermore, if every xv-path also contains y, then x and y are always contained in a common edge of every uv-path. In the latter case x of course always contained in the yv-subpath. Now suppose there are xv-path that do not contain y. We can then subdivide every simple uv-path into an ux-path and an xv-path, and further subdivide the ux-path into a uy-path and yx-path. If x and y always appear in the same edge along P, we can argue as above. Otherwise, x is always contained in the yv-subpath of every uv-path. Thus x always separates y and v. □

From the Lemma 4.8, we propose the following axiom for a general transit function as:

(b3') If x, y, u, v are distinct, $x \in R(u,v)$, and $y \in R(u,x)$ then $y \in R(x,v)$.

Proposition 4.9. *The transit function C of a hypergraph satisfies* (t1)*,* (t2)*,* (b2)*,* (b3')*,* (m)*, and* (U)*.*

Proof. The general axioms (t1) and (t2) are already discussed in the previous section. Lemma 4.4 can be translated to "$x \in C(u,v)$ and $y \in C(u,x)$ implies $y \in C(u,v)$" which is equivalent to "$x \in C(u,v)$ implies $C(u,x) \subseteq C(u,v)$", i.e. axiom (b2). Axiom (b3') is simple rewording of Lemma 4.8. Axiom (U) holds due to Corollary 4.7. Now axiom (m) follows from Corollary 4.7, for $x, y \in C(u,v)$ implies that $C(u,x) \cup C(x,v) = C(u,v)$. Therefore $y \in C(u,x)$ or $y \in C(x,v)$. W.l.g, assume that $y \in C(u,x)$.

So invoking Corollary 4.7 to $C(u,x)$, we get $C(u,y) \cup C(x,y) = C(u,x)$. That is $C(u,y) \cup C(x,y) \cup C(x,v) = C(u,v)$, which proves that $C(x,y) \subseteq C(u,v)$. Similarly it follows that $C(x,y) \subseteq C(u,v)$, if $y \in C(x,v)$ and hence C satisfies (m). □

Suppose u and v are not adjacent and separated by the strong cut vertex x. Then there are (simple) uv-paths containing a ux-path P such that no edge in P contains x, u, and v, and thus $x \in C(u,v)$, $x \neq u, v$ implies $u \notin C(x,v)$ or $v \notin C(u,x)$. Now suppose u, v are adjacent. In contrast to graphs we cannot conclude $C(u,v) = \{u,v\}$. Denoting by E_{uv} the set of all edges appearing in at least one uv-path. Then every xy-path contains an edge containing x and an edge containing y.

Write $K_u := \bigcap\{e \in E_{uv} | u \in e\}$ and $K_v := \bigcap\{e \in E_{uv} | v \in e\}$. Then in particular $u \in V(K_u)$ and $v \in V(K_v)$, i.e., these sets are always non-empty. Note that for each $x \in C(u,v)$ we can also consider $K_x := \bigcap\{e \in E_{uv} | x \in e\}$. Of course, $x \in V(K_x)$, and $V(K_x) \subseteq C(u,v)$. Furthermore, if $x \in V(K_y)$ and $y \in V(K_x)$, then $y \in C(u,x)$ and $x \in C(y,v)$ as well as $y \in C(x,v)$ and $x \in (u,y)$.

For instance, if $e = \{u,x,v\}$ is the only path from u to v, then $C(u,v) = \{u,v,x\}$. Furthermore, if x is not contained in any other edge, we have $x \in C(u,v)$ but $C(u,x) = C(x,v) = C(u,v)$. Hence we do not seem to have an analog of axiom (b1). By the same argument, (b4) and (b3) do not hold.

Finally, we have the following remark for an arbitrary transit function R

Remark 4.10. The axioms (b2), (b3'), and (U*) together imply (U) for an arbitrary transit function R as combining (b3') and (U*) yields "$x \in R(u,v)$ implies $R(u,v) \subseteq R(u,x) \cup R(x,v)$", from which (U) is obtained by using (b2) to establish $R(u,x) \subseteq R(u,v)$ and $R(x,v) \subseteq R(u,v)$.

Problem 4.11. An interesting problem on the cut-vertex transit function of a hypergraph is whether there exists an axiomatic chracterization of C similar to the case of the cut-vertex transit function of a graph.

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