Connectivity Spaces

Bärbel M. R. Stadler and Peter F. Stadler

Abstract. Connectedness is a fundamental property of objects and systems. It is usually viewed as inherently topological, and hence treated as derived property of sets in (generalized) topological spaces. There have been several independent attempts, however, to axiomatize connectedness either directly or in the context of axiom systems describing separation. In this review-like contribution we attempt to link these theories together. We find that despite differences in formalism and language they are largely equivalent. Taken together the available literature provides a coherent mathematical framework that is not only interesting in its own right but may also be of use in several areas of computer science from image analysis to combinatorial optimization.

Mathematics Subject Classification (2000). 54D05, 54E05.

Keywords. Connectedness, generalized closure function, generalized topology, fitness landscapes, additivity.

1. Introduction

Connectedness is a fundamental property of objects and thus plays a key role in particular in devising computational models for them. In topology it was studied already in the early 1900s by Hausdorff, Riesz, and Lennes (see [1] for a historic perspective on this most early work). Topological spaces (and their generalizations known as closure spaces) come endowed with a natural concept of topological connectedness that is usually expressed in terms of a separation relation: two sets A and B are separated if $(A \cap c(B)) \cup (c(A) \cap B) = \emptyset$, where $c(\,.\,)$ denotes the closure. The Hausdorff-Lennes condition stipulates that a set is connected if it cannot be partitioned into two non-empty separated subsets. Connectedness thus is usually treated as a derived property of spaces that are defined in terms of notions of boundary, closure, interior, or neighborhood. An early exposition

This work was supported in part by the *Deutsche Forschungsgemeinschaft* (DFG) Project STA850/11-1 within the EUROCORES Programme EuroGIGA (project GReGAS) of the European Science Foundation.

of connectedness is [2]. From a category theory point of view, connectedness is instead defined in terms of continuity, using a classical theorem from topology as definition: X is ζ-connected if and only if every ζ-continuous function $f : X \to \nvDash$ is constant, where $\nvDash$ is the discrete space with two points [3, 4, 5].

Generalized topologies $(X, \mathcal{O})$ in the sense of Császár [6] consist of a set system $\mathcal{O} \subseteq 2^X$ that is closed under arbitrary unions, i.e., if $O_\iota \in \mathcal{O}$ for $\iota \in I$, then $\bigcup_{\iota \in I} O_i \in \mathcal{O}$. The elements of $\mathcal{O}$ serve as generalization of open sets. In this setting a space X is $\mathcal{O}$-connected if there are no two disjoint non-empty open sets $O_1, O_2 \in \mathcal{O}$ such that $O_1 \cup O_2 = X$. A variety of derived generalized open set systems can be constructed in terms of the closure and interior operators on $\mathcal{O}$, thus giving rise to different flavors of connectedness, see e.g. [7, 8, 9] for a systematic analysis.

In the 1940s, several authors investigated connectedness as the fundamental concept of a topological theory starting from axioms for separation between sets instead of deriving separation from another topological structure [10, 11, 12]. It was soon recognized this leads to theories that are substantially more general than topological spaces [12, 13, 14]. In particular, there are natural connectivity structures that do not coincide with the connected sets of any topological space (or generalized closure space) [15]. The notion of separation is intimately related to that of *proximity* [16]. A set A is proximally-connected in a proximity space (X, δ) if, for any two non-empty sets A', A'' with $A' \cup A'' = A$ it holds that $A' \delta A''$ [3], i.e., if A cannot be separated into a pair of *far sets*, see [17] for some further developments.

Starting with the 1980s, several authors have begun again to investigate systems of connected sets either in their own right [18, 19], or motivated by particular applications. A close connection between knot theory and finite connectivity spaces has been explored in [20]. Connectivity is also an important concept in digital image analysis and has become a focus in Mathematical Morphology starting with the work of Serra, Ronse, and collaborators [21, 22] and centers on filter operations, called openings or closings, that remove grains or fill in pores [23] and provide an abstract definition of connected components. Several natural constructions in this context lead to collections of "connected sets" that are not derivable from topologies:

- The standard notion of *connectivity of graphs*, for instance, is only topological in a relaxed sense. It is derived from the Hausdorff-Lennes condition on pretopologies, which lack idempotence of the close operator $c(\,.\,)$ rather than topological spaces [24].
- Different notions of connectivity on hypergraphs have been investigated. Most naturally, each hyperedge is considered as a connected set. An interesting alternative notion that is equivalent to connectedness for graphs but in general not on hypergraphs is *partition connectedness.* A hypergraph H is called *partition connected* [25] if every partition of its vertex set into t classes is such that there are at least $t-1$ hyperedges that intersect at least two classes. As

there are a variety of notions of paths in hypergraphs, there are also multiple nonequivalent concepts of path connectedness.

- A set A is *arc-wise connected* if for every pair $x, y \in A$ there is a continuous (w.r.t. a given topology) function $\gamma : [0, 1] \to A$ such that $\gamma(0) = x$, $\gamma(1) = y$, and γ is a homeomorphism between the unit interval $[0, 1]$ of $\mathbb{R}$ and its image. The closure of the "topologist's sine curve", i.e., the set $S := \{(x, \sin x^{-1})|0 < x \leq 1\} \cup \{(0, y)| -1 \leq y \leq 1\}$, for instance, is connected w.r.t. to the standard topology of the plane, but it is not arc-wise connected. A theorem by Mròwka [15, Thm.1] demonstrated that the collection $\mathcal{A}(\mathbb{R}^2)$ of all arc-wise connected sets in the plane cannot be obtained as the connected sets w.r.t. any topology on $\mathbb{R}^2$.
- A set A is polygonal line connected in $\mathbb{R}^n$, $A \in \mathcal{P}(\mathbb{R}^n)$ if every pair of points $x, y \in A$ can be joined by a polygonal line entirely contained in A [26]. In this setting, objects are not connected if they have thin, curved parts.
- The family of all subsets of a Euclidean space that are not linearly separable forms a connectivity space [26].
- *Dilation connectivity* is defined in terms of one or more structuring connected sets Q [27]. Denote by $Q\xi$ the translation of Q to the origin ξ. The dilation of a set A by Q is the set of all points ξ such that $Q\xi \cap A \neq \emptyset$. The dilation connectivity provides a direct link to mathematical morphology [28, 29, 30]. Connected sets with respect to the dilation connectivity have an interpretation as clustering of A.

Connectedness, and in particular the notion of connected components, are of key interest in topological approaches to image analysis. It is of eminent practical importance for image filtering and segmentation, image compression and coding, motion analysis, and pattern recognition, see e.g. [31, 32, 33, 34]. The central role of connectivity in this area derives from the idea that *the connected components of an object are "non-overlapping" and their "union" reconstitutes the original object* [33], providing a solid theoretical foundation for partitioning, i.e., segmenting, objects.

More recently, the investigation of generalized topologies associated with chemical reaction networks has lead to "constructive connectedness" as a more natural notion of connectivity of chemical spaces than the Hausdorff-Lennes connectivity on the same neighborhood space [35]. The motivation here comes from the investigation of large chemical reaction networks, i.e., directed hypergraphs. In this context the closure $c(A)$ of a set of chemical compounds is the set of chemical substances that can be formed by chemical reactions whose educts are taken from A. Naturally A and B are separated whenever all reactions that can take place in $A \cup B$ are also feasible in A alone or in B alone, see Fig 1. This notion of connectedness does *not* match the usual concepts of connectedness in hypergraphs.

An axiomatic approach to connectivity is also of interest in the context of fitness landscapes, i.e., functions $f : X \to \mathbf{R}$, where $\mathbf{R}$ is some well-ordered set, X is the underlying search space, and f is a fitness or cost function. While X is

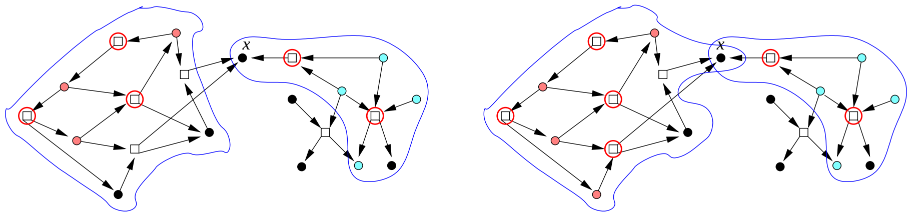

FIGURE 1. Example of productive connectivity in a chemical reaction network. The directed hypergraph X is represented as a bipartite digraph such that each $\bullet$ denotes a chemical compound (points of X) and each $\square$ is a reaction. A reaction is "active" in a set Z of compounds (here marked by red and cyan, round vertices) if all its inputs are contained in Z. All active reactions for Z are marked by circles. The two panels show slightly different sets Z. The closure function is defined such that $c(Z)$ contains all points of Z together with the products of all reactions that are active for Z. $c(Z)$ is indicated by blue outlines. As defined in [35], (A, B) is a *productive separation* of Z if, for all subsets $Z' \subseteq Z$ holds (i) $c(A \cap Z') \cap B = \emptyset$ and $c(B \cap Z') \cap A = \emptyset$, and (ii) $c(Z') = c(Z' \cap A)$ and $c(Z') = c(Z' \cap B)$. Choose A as the red and B as the blue vertices of Z. Since c is enlarging by construction, condition (i) reduces to the Hausdorff-Lennes condition $c(A) \cap B = c(B) \cap A = \emptyset$. It is clearly satisfied here. Condition (ii) also holds since there is no active reaction for Z that has inputs in both A and B. Note that from the point of view of active reactions (circled reaction nodes) one might want to consider the right hand side as connected since their sets of reaction products share x.

finite (but usually unmanageably large) in combinatorial optimization problems, X is usually taken to be a continuum in the field of evolutionary computation. In both settings, coarse grained representations (such as barrier trees [36]), as well as elaborate stochastic models of optimization algorithms make prominent use of the connected components of "level sets" $F_h = \{x|f(x) \leq h\}$. A closer inspection shows that the only intrinsic structure of the search space X that is actually used in this context is the connectedness of its subsets. This is most transparent in Trouvé's "cycle decomposition" of the state space [37] in the theory of simulated annealing. Here, the connected components of the restriction of the search space to solutions with a prescribed maximum cost play the key role, see Fig. 2. In most applications of this type, from simulated annealing to RNA landscapes [24], an additive (graph-like) connectivity structure is used. In [36], the connectivity structure

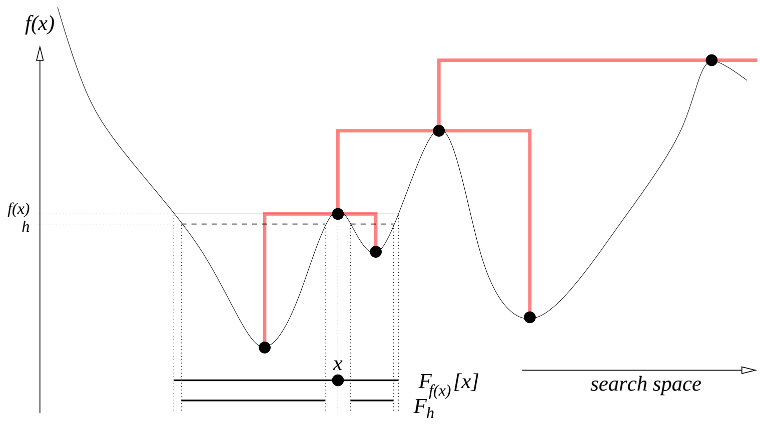

FIGURE 2. A combinatorial landscape and its barrier tree. A landscape consists of a search space X and a cost function $f : X \to \mathbb{R}$. A crucial concept in analyzing landscapes are the connected components of the sets $F_h := \{x \in X | f(x) \leq h\}$. Denote by $F_h[x]$ the connected component of F_h that contains x. Local minima and saddle points are most conveniently defined in terms of connected sets: x is a local minimum if $y \in F_{f(x)}[x]$ implies $f(y) = f(x)$. A saddle point x is characterized by the fact that its connected component breaks up as soon as the cost cut-off h falls below $f(x)$, i.e., if there is an $\epsilon > 0$ such that $F_h[x] \cap F_{h'}$ is not connected for all h, h' with $f(x) - \epsilon < h' < f(x) < h < f(x) + \epsilon$. The barrier tree [38] has the local minima as its leaves and the saddle points as its interior vertices. Note that saddle points appear as local maxima of f only in two-dimensional drawings, i.e. when X is represented by a single coordinate axis.

of a discrete search space is discussed that is implied by a genetic algorithm: For a population A, its closure $c(A)$ is understood as the set of all possible offspring that can be generated from A, similar to chemical spaces discussed above. The resulting generalized topology then induces a natural notion of connectedness.

The connectedness of the set of optimal solutions in a multi-objective optimization problem is known to have an impact on the performance of heuristics [39]. Minimax theorems for functions of the form $g : X \times Y \to \mathbb{R}$ also involve the connectedness of level sets in one variable. This generates a separation structure in the sense of Wallace rather than a topology on the sets X and Y [40].

It is the purpose of this contribution to summarize elementary results on connectivity spaces and their associated separation relations. Much of the material compiled here is "mathematical folklore" and many results have already been obtained by others in one form or the other. Since we strive to present the connections between the different formalisms in their most general form, we nevertheless include simple proofs of many statements that have been published only in a less general setting or for which we could not find an easily accessible proof in the literature. We deliberately concentrate on simple, very basic properties of connectivity structures in an attempt to connect independent lines of reasoning and results that have been scattered in the literature.

2. General Connectivity Spaces

2.1. Basic Definitions

Throughout this contribution we are interested in connectedness structures on an arbitrary set X. The most direct approach is to specify axioms for a collection $\mathcal{C} \subseteq 2^X$ of *connected subsets* of X.

Definition 2.1. A connectivity space is a pair $(X, \mathcal{C})$ with $\mathcal{C} \subseteq 2^X$ such that

(c0) $\emptyset \in \mathcal{C}$

(c1) $Z_\imath \in \mathcal{C}$ for all $\imath \in I$ and $\bigcap_{\imath\in I} Z_\imath \neq \emptyset$ implies $\bigcup_{\imath\in I} Z_\imath \in \mathcal{C}$

Connectivity spaces [21, 18, 20] have also been termed "connective spaces" or "c-spaces" in [19], and "partial connection" in [41]. An equivalent definition, using the name "connectivity system" was used in [42].

Consider an arbitrary set $Z \in \mathcal{C}$ and let $\mathcal{B}_Z$ be a subset of $\mathcal{C}$ such that (i) $Z \in \mathcal{B}_Z$ and (ii) $Z' \in \mathcal{B}_Z$ implies $Z' \cap Z \neq \emptyset$.

Fact 2.2. $\mathcal{C}$ *satisfies axiom (c1) if and only if the following holds:*

(c1') *Let $Z \in \mathcal{C}$ and $\mathcal{B}_Z \subseteq \mathcal{C}$ such that (i) $Z \in \mathcal{B}_Z$ and (ii) $Z' \in \mathcal{B}_Z$ implies $Z' \cap Z \neq \emptyset$. Then $\bigcup_{Z'\in\mathcal{B}_Z} Z' \in \mathcal{C}$.*

Proof. Suppose (c1) holds and Z, Z' and $\mathcal{B}_Z$ is defined as in (c1'). Then $Z' \cup Z \in \mathcal{C}$ for all $Z' \in \mathcal{B}_Z$. Furthermore $\bigcup_{Z'\in\mathcal{B}_Z} Z' = \bigcup_{Z'\in\mathcal{B}_Z}(Z' \cup Z)$. The latter sets are connected and their intersection contains Z, hence their union is connected as well by axiom (c1). The converse is obvious. □

2.2. Connected Components

The concept of connected components is a key ingredient of any theory of connectivity.

Definition 2.3. For every $A \subseteq X$ and every $x \in X$ the set

$$A[x] = \bigcup \left\{ A' \subseteq X \,\middle|\, A' \subseteq A,\, x \in A',\, A' \in \mathcal{C} \right\} \tag{2.1}$$

is called the *connected component* of $x \in A$.

By definition $A[x] = \emptyset$ if $x \notin A$. Furthermore $A[x] \in \mathcal{C}$ as a direct consequence of axiom (c1) for non-empty $A[x]$ and of axiom (c0) for $A[x] = \emptyset$. It is important to note that $x \in A$ does not guarantee that $A[x]$ is non-empty.

Fact 2.4. *Given an arbitrary collection $\mathcal{B} \subseteq 2^X$ of connected sets there is a unique minimal collection $\mu'(\mathcal{B}) \subseteq 2^X$ such that $\mathcal{B} \subseteq \mu'(\mathcal{B})$ and $\mu'(\mathcal{B})$ satisfies (c0) and (c1).*

This observation has been made repeatedly in the literature for more restricted types of connectivity spaces, see e.g. [19]. A complete discussion in full generality can be found in [41]. By construction $\mu' : 2^{2^X} \to 2^{2^X}$ is idempotent, i.e., $\mu'(\mu'(\mathcal{B})) = \mu'(\mathcal{B})$ and expanding, i.e., $\mathcal{B} \subseteq \mu'(\mathcal{B})$. $\mathcal{B}$ is called a *basis* of the connectivity space $\mu'(\mathcal{B})$. Furthermore, we say that $\mathcal{B}$ is *complete* if $\mathcal{B} = \mu'(\mathcal{B})$, i.e., if the basis already satisfies (c0) and (c1).

The connected sets of $\mu'(\mathcal{B})$ can be characterized directly in terms of "chaining" [41], see also [26] for the analogous construction in a more restrictive setting: $A \in \mu'(\mathcal{B})$ if and only if any two points $x, y \in A$ can be joined by a sequence of connected sets $Z_i \in \mathcal{B}$, $Z_i \subseteq A$, $0 \le i \le n$ such that $x \in Z_0$, $y \in Z_n$, and $Z_{i-1} \cap Z_i \neq \emptyset$ for $0 < i \le n$.

An alternative construction, first described in [43], see also [44, Thm.3], uses transfinite induction to produce the connected components: For given $A \in 2^X$ and $x \in A$ let $A_0[x] = \bigcup\{Z \in \mathcal{B} | Z \subseteq A,\, x \in Z\}$. Then define recursively $A_k[x] = A_{k-1}[x] \cup \bigcup\{Z \in \mathcal{B} | Z \subseteq A,\, Z \cap A_{k-1}[x] \neq \emptyset\}$ and set $A^*[x] = \bigcup_{k=0}^{\infty} A_k[x]$. By (c1) $A^*[x] \in \mu'(\mathcal{B})$ and $A^*[x] = A$ for $A \in \mathcal{B}$ and $x \in A$. Thus $\mu'(\mathcal{B}) = \{A^*[x] | A \in 2^X,\, x \in A\}$ and the $A^*[x]$ are indeed the connected components of A w.r.t. $\mu'(\mathcal{B})$. Note that $A[x]$ can be empty since $A^*[x] = \emptyset$ if and only if $A_0[x] = \emptyset$.

In this most general setting, therefore, the set system $\{A[x] | x \in A\}$ does not define a partition of A but only a partial partition [41], and may even consist of the empty set only. We write $\mathring{A} = \{x \in A | A[x] = \emptyset\}$ for the part of A not covered by connected components. If $\mathcal{C}$ satisfies (c0) and (c1), $\mathring{A}$ is the set of all points in A that are not contained in any connected subset of A. An example of a connectivity space in which not all points are covered by connected components is shown in Fig. 3.

For later reference we note the following simple

Fact 2.5. *Let C be a connected component of A and suppose $C \subseteq B \subseteq A$. Then C is a connected component of B.*

Proof. By construction C is connected and there is no connected subset of A that intersects both C and $A \setminus C$. Hence no such set exists in $B \subseteq A$, i.e., C is a maximal connected subset of B. □

2.3. Connectivity Openings

An alternative starting point to the theory of connectivity spaces are the properties of connected components. This avenue was explored e.g. in [21, 22, 26, 41, 45].

Definition 2.6. A "connectivity opening" is a map $\gamma : X \times 2^X \to 2^X : (x, A) \mapsto A[x]$ that satisfies for all $x \in X$ and all $A, B \in 2^X$ the following axioms:

(o0) $x \notin A$ implies $A[x] = \emptyset$.
(o1) $A[x] \subseteq A$.

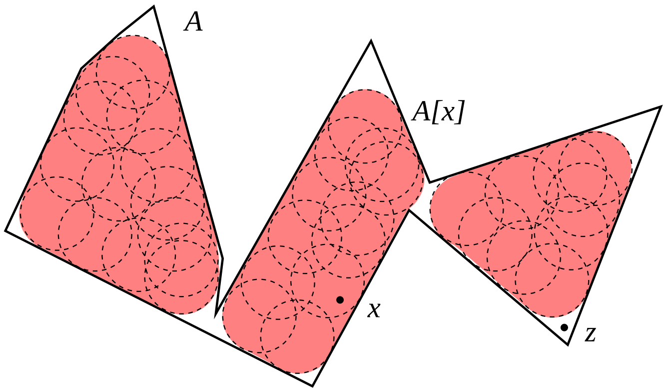

Figure 3. Example of a connective space with non-connected points. Every unit circle (dashed) that can be fit into A (outlined in black) is connected. By (c1), so is the union of embedded cycles, shown here as three disjoint shaded areas that constitute the connected components of A. A point x located within a (connected) circle and its connected component $A[x]$ are marked. In contrast, z is not located inside a circle, i.e., $A[z] = \emptyset$, as for any point located in the white parts of A, which together represent $\mathring{A}$.

(o2) $A \subseteq B$ implies $A[x] \subseteq B[x]$.
(o3) $(A[x])[x] = A[x]$.
(o4) If $y \in A[x]$ then $A[y] = A[x]$.

For later reference we note that (o2) is equivalent to

$$A[x] \cup B[x] \subseteq (A \cup B)[x] \tag{2.2}$$

Theorem 2.7. [41, Thm.21] *There is one-to-one correspondence between systems of connected components satisfying axioms (o0) to (o4) and connectivity spaces satisfying (c0) and (c1) given by Definition* 2.3 *and*

$$\mathcal{C} = \left\{ Z \in 2^X \middle| Z = A[x],\, A \subseteq X,\, x \in X \right\} \tag{2.3}$$

The structure of connected components as partial partitions forms the basis of a systematic investigation of the lattice of partial partitions in [45]. This work identifies a particular type of block-splitting operators that produce exactly the connectivity openings. This order and lattice-theoretic angle is developed further in [46, 47].

3. Separation Spaces

3.1. Separations of Connectivity Spaces

Definition 3.1. Let $\mathcal{C} \subseteq 2^X$ be an arbitrary collection of connected sets. A pair (A, B) is $\mathcal{C}$-*separating* if for every connected subset $Z \in \mathcal{C}$ with $Z \subseteq A \cup B$ holds $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$.

The intuition behind this definition is that the pair (A, B) does not "separate" any connected set. By construction every non-empty connected subset $Z \subseteq A \cup B$ lies either in $A \setminus B$ or in $B \setminus A$. We write $\mathfrak{S}_{\mathcal{C}}$, or when the context is clear, simply $\mathfrak{S}$ for the collection of all separations w.r.t. to a given collection of connected sets $\mathcal{C}$.

Recall that $\mathring{Y}$ is the set of points in $Y \subseteq X$ that do not belong to any connected component of Y, i.e., a point $y \in \mathring{Y}$ is separated from every subset of Y including $\{y\}$ itself. More formally, if there is a separation $(A, B) \in \mathfrak{S}$ with $y \in Y \subseteq A \cup B$ and $y \in A \cap B \neq \emptyset$, then $y \in \mathring{Y}$. Conversely, for $y \in \mathring{Y}$ we have $(\{y\}, Y') \in \mathfrak{S}$ for all $Y' \subseteq Y$. In particular, $(\mathring{Y}, \mathring{Y}) \in \mathfrak{S}$ and $(\{y\}, \{y\}) \in \mathfrak{S}$ for all $y \in \mathring{Y}$. Thus we can express $\mathring{Y}$ in terms of the separation $\mathfrak{S}$ associated with $\mathcal{C}$ as follows:

$$\mathring{Y} = \{y \in Y | \exists (A, B) \in \mathfrak{S} \text{ such that } Y \subseteq A \cup B,\ y \in A \cap B\} \tag{3.1}$$

The separation $\mathfrak{S}$ defined by a collection $\mathcal{C}$ of connected sets already has several interesting properties:

Theorem 3.2. *The separation $\mathfrak{S}$ w.r.t. $\mathcal{C}$ satisfies*

(S0) $(A, \emptyset) \in \mathfrak{S}$
(S1) $(A, B) \in \mathfrak{S}$ *implies* $(B, A) \in \mathfrak{S}$.
(S2) $(A, B) \in \mathfrak{S}$, $A' \subseteq A$, *and* $B' \subseteq B$ *implies* $(A', B') \in \mathfrak{S}$.
(SR0) $Y \setminus \mathring{Y} \subseteq A \cup B \subseteq Y$ *and* $(A, B) \in \mathfrak{S}$ *implies* $\left(A \cup \mathring{Y}, B \cup \mathring{Y}\right) \in \mathfrak{S}$.
(SR1) $(A, B) \in \mathfrak{S}$ *and* $(A \cup B, C) \in \mathfrak{S}$ *implies* $(A, B \cup C) \in \mathfrak{S}$.
(SR2) *If* $(A_i, B_i) \in \mathfrak{S}$ *and* $A_i \cup B_i = Y \subseteq X$ *for* $i \in I$ *then* $\left(\bigcap_{i \in I} A_i, \bigcup_{i \in I} B_i\right) \in \mathfrak{S}$.

Proof. (S0) and (S1) follow trivially from the definition. For every $Z \in \mathcal{C}$ such that $Z \subseteq A \cup B$ we have $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$. Thus $Z \cap A' \subseteq Z \cap A = \emptyset$ or $Z \cap B' \subseteq Z \cap B = \emptyset$ holds for all $A' \subseteq A$ and $B' \subseteq B$ in particular whenever $Z \subseteq A' \cup B' \subseteq A \cup B$. Thus $(A', B') \in \mathfrak{S}$, and (S2) is satisfied.
Suppose $Y \setminus \mathring{Y} \subseteq A \cup B$. By definition $\mathring{Y}$ does not intersect any connected subsets of Y. Thus, for all $Z \in \mathcal{C}$, $Z \subseteq Y$ holds $A \cap Z = \emptyset$ if and only if $(A \cup \mathring{Y}) \cap Z = \emptyset$. Since all connected subsets of Y are contained in $Y \setminus \mathring{Y}$, we can conclude for every separation $(A, B) \in \mathfrak{S}$ with $Y \setminus \mathring{Y} \subseteq A \cup B$ that $(A \cup \mathring{Y}, B \cup \mathring{Y}) \in \mathfrak{S}$.
Every set $Z \in \mathcal{C}$ with $Z \subseteq A \cup B \cup C$ satisfies $Z \cap (A \cup B) = \emptyset$ or $Z \cap C = \emptyset$ because $(A \cup B, C) \in \mathfrak{S}$. If $Z \cap C = \emptyset$ then $Z \subseteq A \cup B$ and the assumption $(A, B) =\in \mathfrak{S}$ implies $Z \cap A = \emptyset$ or $Z \cap B = Z \cap (B \cup C) = \emptyset$. Otherwise, $Z \cap (A \cup B) = \emptyset$ implies in particular $Z \cap A = \emptyset$. Thus, in both cases $Z \cap A = \emptyset$ or $Z \cap (B \cup C) = \emptyset$. Since this is true for all $Z \in A \cup B \cup C$, we have $(A, B \cup C) \in \mathfrak{S}$.

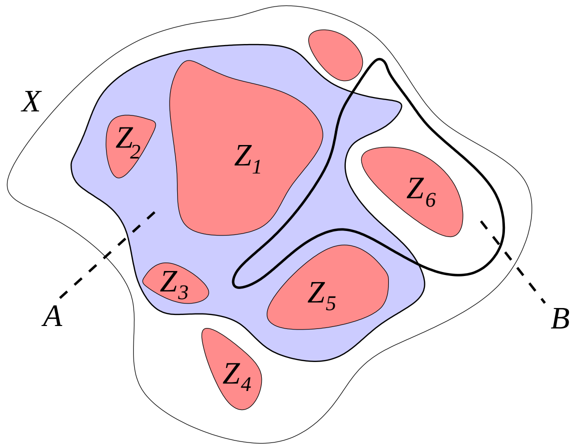

FIGURE 4. Symmetric separations and connected sets. The pair (A, B) separates X since no connected set contained in $A \cup B$ (Z_1 through Z_6) intersects both A and B. Note that the intersection of A and B need not be empty. However, $A \cap B$ must not contain a connected set.

Consider an arbitrary $Z \in \mathcal{C}$ with $Z \subseteq Y = A_i \cup B_i$ for some $Y \subseteq X$ and $(A_i, B_i) \in \mathfrak{S}$. If $Z \cap A_i = \emptyset$ for some $j \in I$ then $Z \cap \bigcap_{i \in I} A_i = \emptyset$. In the remaining case, $Z \cap A_i \neq \emptyset$ and thus $Z \cap B_i = \emptyset$ for all $i \in I$. This implies $\bigcup_{i \in I}(Z \cap B_i) = Z \cap \left(\bigcup_{i \in I} B_i\right) = \emptyset$. Thus (SR2) holds. □

A set of pairs $\mathfrak{S} \subseteq X \times X$ is called *grounded* if it satisfies (S0), *symmetric* if (S1) holds, and *hereditary* if (S2) is satisfied. $\mathfrak{S}$ is a *separation* if it satisfies (S0) and (S2). Separations are equivalent to so-called semi-topogenous orders on 2^X by virtue of $(A, B) \in \mathfrak{S}$ iff $A \prec X \setminus B$ [48, 49]. Non-symmetric separations have been considered as an alternative basis of topology and in the context of quasi-proximities [50]. Since connectivity is inherently symmetric, however, there is nothing to be gained for our purposes by dropping the symmetry axiom (S1), hence we will only consider symmetric separations throughout this contribution. Symmetric separation spaces have already been studied in some detail in [12, 13, 14, 43, 49]. The converse of separation is *proximity* or nearness. The concept was introduced by Frigyes Riesz already in 1909 [51]. Since proximities have been studied mostly with much stronger axiom systems than the ones of interest here, we will return to this point of view only later, see Section 7.4.

3.2. Ronse's Axioms

Properties (SR1) and (SR2) were introduced in a somewhat different and a slightly less general form by Ronse [26]. The following discussion heavily borrows from

[26] to establish analogous results for connectivity spaces in full generality. Most prominently we add property (SR0) here to deal with those points that do not belong to any connected component.

The following property effectively describes when a symmetric separation has a structure that conforms to the idea of connected components.

(SR3) Every set $Y \subseteq X$ can be partitioned into a set Y° and a collection Y_j, $j \in J$, such that
  (i) For all $y \in Y^\circ$ there is $(A, B) \in \mathfrak{S}$ such that $y \in A \cap B$ and $A \cup B = Y$; For all $(A, B) \in \mathfrak{S}$ with $Y_j \subseteq A \cup B$ we have $A \cap Y_j = \emptyset$ or $B \cap Y_j = \emptyset$;
  (ii) For every subset $K \subseteq J$ we have

$$\left(\bigcup_{j \in K} Y_j \cup Y^\circ, \bigcup_{j \in J \setminus K} Y_j \cup Y^\circ \right) \in \mathfrak{S}$$

We will show that the sets Y_j play the role of the connected components of Y, while Y° comprises the points of Y not contained in any connected component of Y.

Lemma 3.3. *If $\mathfrak{S}$ is a symmetric separation satisfying (SR0), (SR1), and (SR2) then (SR3) also holds.*

Proof. Consider $Y \subseteq X$ and define $\mathcal{Y} = \{(U, V) | U \cup V = Y \text{ and } (U, V) \in \mathfrak{S}\}$. We define $Y(p) = \emptyset$ if there is $(U, V) \in \mathcal{Y}$ such that $p \in U \cap V$ and set $Y^\circ := \{p \in Y | Y(p) = \emptyset\}$. Otherwise we can, by exploiting symmetry, rename all pairs $(U_i, V_i) \in \mathcal{Y}$ such that $p \in U_i$ and $p \notin V_i$ and define $Y(p) = \bigcap_i U_i$. Since $Y \setminus U_i \subseteq V_i$ must hold for all $(U_i, V_i) \in \mathcal{Y}$, we have $Y \setminus Y(p) \subseteq \bigcup_i V_i$ and thus by (SR2) $(Y(p), Y \setminus Y(p)) \in \mathfrak{S}$ and hence also $(Y(p), Y \setminus Y(p)) \in \mathcal{Y}$.

Now suppose $q \in Y(p)$ and $Y(q) = \emptyset$. By construction, then, there is $(U, V) \in \mathcal{Y}$ with $q \in U \cap V$, $p \in U$, and $p \notin V$ since $Y(p) \neq \emptyset$. By (S2), $(U \setminus V, V) \in \mathcal{Y} \subseteq \mathfrak{S}$, and hence $p \in U \setminus V$ and $q \notin U \setminus V$. Since the intersection in the definition of $Y(p)$ also runs over $U \setminus V$, we arrive at the contradiction $q \notin Y(p)$. Thus $Y(p) \cap Y^\circ = \emptyset$.

By definition of $Y(p)$ and Y° we have $Y^\circ \cup \bigcup_{p \in Y} Y(p) = Y$. Now suppose $Y(p) \neq \emptyset$ and $Y(q) \neq \emptyset$. As shown above, $(Y(p), Y \setminus Y(p)) \in \mathcal{Y}$ and $(Y(q), Y \setminus Y(q)) \in \mathcal{Y}$, whence $Y(q) \cap Y(p) = \emptyset$ or $Y(q) \cap (Y \setminus Y(p)) = \emptyset$; and $Y(p) \cap Y(q) = \emptyset$ or $Y(p) \cap (Y \setminus Y(q)) = \emptyset$. Thus $Y(p) \cap Y(q) = \emptyset$, or both $Y(q) \cap (Y \setminus Y(p)) = \emptyset$ and $Y(p) \cap (Y \setminus Y(q)) = \emptyset$ must hold, whence $Y(q) \subseteq Y(p)$ and $Y(p) \subseteq Y(q)$, i.e., $Y(p) = Y(q)$.

Thus the $Y(p)$ together with Y° form a partition of Y. Denote by Y_j, $j \in J$, the collection of the non-empty $Y(p)$, and suppose there is $(P, Q) \in \mathfrak{S}$ such that $Y_j \subseteq P \cup Q$ and $Y_j \cap P \neq \emptyset$ and $Y_j \cap Q \neq \emptyset$. Then $(Y_j \cap P, Y_j \cap Q) \in \mathfrak{S}$ by (S2). On the other hand, we have already shown that $(Y_j, Y \setminus Y_j) \in \mathfrak{S}$. Following [26] set $A := P \cap Y_j$, $B := Q \cap Y_j$, and $C := Y \setminus Y_j$ we find $(A, B) \in \mathfrak{S}$ and $(A \cup B, C) \in \mathfrak{S}$. By (SR1) we therefore have $(P \cap Y_j, Y \setminus Y_j \cup (Q \cap Y_j)) \in \mathcal{Y} \subseteq \mathfrak{S}$. This contradicts the minimality of Y_j unless $P = Y_j$. Since $Y_j \cap Y^\circ = \emptyset$ we have $Y_j \cap Q = \emptyset$, a contradiction. Hence the partition of Y satisfies statement (i) of (SR3).

From $(Y_j, Y \setminus Y_j) \in \mathfrak{S}$ for $j \in J$ we obtain by (SR2) $\left(\bigcap_{i\in I}(Y \setminus Y_i), \bigcup_{i\in I} Y_i\right) \in \mathfrak{S}$ for all $I \subseteq J$. Because $Y_k \subseteq Y \setminus \bigcup_{i\in I} Y_i$ for all $k \in K \subseteq J \setminus I$ we have by (S2) also $\left(\bigcup_{i\in I} Y_i, \bigcup_{i\in K} Y_i\right) \in \mathfrak{S}$ whenever $K \cap I = \emptyset$.

Property (SR0) now immediately implies (ii), i.e., (SR3) holds. □

Lemma 3.4. *Let $\mathfrak{S}$ be a symmetric separation satisfying (SR3) and let $U \subseteq Y \subseteq X$. Then $(U, Y \setminus U) \in \mathfrak{S}$ implies that $U = U^\circ \cup \bigcup Y_i$, where $U^\circ \subseteq Y^\circ$ and the Y_i are classes of the partition defined by (SR3). Furthermore, U can be replaced by $U \cup Y^\circ$.*

Proof. $(U, Y \setminus U) \in \mathfrak{S}$ implies $Y_j \cap U = \emptyset$, i.e., $Y_j \subseteq Y \setminus U$, or $Y_j \cap (Y \setminus U) = \emptyset$, i.e., $Y_j \subseteq U$. Thus U and $Y \setminus U$ together contain all Y_j, $j \in J$ and each Y_j is contained in either U or $Y \setminus U$. The remainders of U and $Y \setminus U$ therefore are necessarily subsets of Y°. □

Lemma 3.5. *Let $\mathfrak{S}$ be a symmetric separation satisfying (SR3). Then $\mathfrak{S}$ satisfies (SR0), (SR1), and (SR2).*

Proof. (SR0) follows directly from (ii).

To verify (SR1), consider three not necessarily distinct sets $A, B, C \subseteq X$ with $(A, B) \in \mathfrak{S}$ and $(A \cup B, C) \in \mathfrak{S}$, and let $Y = A \cup B \cup C$. Then by Lemma 3.4 we have the representations $A = \bigcup_{i\in I_A} Y_i \cup A^\circ$, $B = \bigcup_{i\in I_B} Y_i \cup B^\circ$, $C = \bigcup_{i\in I_C} Y_i \cup C^\circ$. Since $(A, B) \in \mathfrak{S}$, property (SR3;i) implies $A \cap Y_j = \emptyset$ or $B \cap Y_j = \emptyset$, i.e., $I_A \cap I_B = \emptyset$. Analogously, we can conclude that $(I_A \cup I_B) \cap I_C = \emptyset$, i.e., the index sets I_A, I_B, and I_C are pairwise disjoint. Thus, by (SR3;ii) we have $\left(\bigcup_{j\in A} Y_j \cup Y^\circ, \bigcup_{j\in I_B \cup I_C} Y_j \cup Y^\circ\right) \in \mathfrak{S}$. Heredity now implies $(A, B \cup C) \in \mathfrak{S}$, i.e., (SR1) holds.

To verify (SR2) let $Y \subseteq X$ and consider a family of pairs $(A_i, B_i) \in \mathfrak{S}$, $i \in I$, with $(A_i \cup B_i) = Y$ such that $\bigcup A_i \neq \emptyset$. From Lemma 3.4 we have, for each $i \in I$, $A_i = A_i^\circ \cup \bigcup_{j\in J_i} Y_j$. Let $K = \bigcap_{i\in I} J_i$ and set $A^\circ = \bigcap_{i\in I} A_i^\circ$. Then $A^\circ \cup \bigcup_{j\in K} Y_j \subseteq \bigcap_{i\in I} A_i$. For every $j \notin K$, there is an $i \in I$ such that $Y_j \in B_i$, i.e., $\bigcup_{j\in J\setminus K} Y_j \subseteq \bigcup_{i\in I} B_i$. Thus $\left(\bigcap_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \in \mathfrak{S}$, i.e., (SR2) holds. □

Taken together we observe that (SR3) is equivalent to (SR0), (SR1), and (SR2). The following observation characterizes the $\mathcal{C}$-separating pairs in terms of connected components. It is again a direct generalization of results by Ronse [26].

Lemma 3.6. *Given a connectivity space $\mathcal{C}$ on X, the corresponding separation space $\mathfrak{S}_\mathcal{C}$ consists of all pairs (A, B) such that every connected component of $A \cup B$ is contained in either A or B.*

Proof. Suppose there is a connected component $Z \subseteq A \cup B$ such that $A \cap Z \neq \emptyset$ and $B \cap Z \neq \emptyset$. Then $(A, B) \notin \mathfrak{S}_\mathcal{C}$. On the other hand, suppose that (A, B) is such that every connected component of $A \cup B$ is contained in either A or B, and let $Z' \in \mathcal{C}$, $Z' \subseteq A \cup B$. Then Z is contained in either A or B since otherwise the connected component Z of $A \cup B$ that contains Z' would not be contained in either A or B. Thus $\mathfrak{S}_\mathcal{C}$ is characterized as claimed. □

3.3. $\mathfrak{S}$-connectedness

Originally conceived by Wallace [12], a notion of connected sets can be obtained in a very natural manner from a collection $\mathfrak{S}$ of pairs of separated sets.

Definition 3.7. Let $\mathfrak{S}$ be a symmetric separation on X. A set $Z \in X$ is called $\mathfrak{S}$-connected if $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$ for all $(A, B) \in \mathfrak{S}$ such that $Z \subseteq A \cup B$.

We write $\mathcal{C}_{\mathfrak{S}}$ for the collection of the $\mathfrak{S}$-connected sets.

Theorem 3.8. *If $\mathfrak{S} \subseteq 2^X \times 2^X$ is symmetric, then the the collection $\mathcal{C}_{\mathfrak{S}}$ of $\mathfrak{S}$-connected sets on X satisfies (c0) and (c1).*

Proof. $\emptyset \in \mathcal{C}_{\mathfrak{S}}$ follows trivially from $A \cap \emptyset = \emptyset$, thus (c0) holds. Now fix a pair $(A, B) \in \mathfrak{S}$ and let $\mathcal{B} \subseteq \mathcal{C}_{\mathfrak{S}}$ be an arbitrary collection of $\mathfrak{S}$-connected sets such that (i) $Z \subseteq A \cup B$ and (ii) there is z such that $z \in Z$ for all $Z \in \mathcal{B}$. W.l.o.g. suppose $z \in A$. Since Z is $\mathfrak{S}$-connected and $z \in A \cap Z$, we have $Z \cap B = \emptyset$, and thus

$$\bigcup_{Z\in\mathcal{B}} (Z \cap B) = \left(\bigcup_{Z\in\mathcal{B}} Z \right) \cap B = \emptyset$$

whenever $\bigcup_{Z\in\mathcal{B}} Z \subseteq A \cup B$. Thus $\bigcup_{Z\in\mathcal{B}} Z$ is $\mathfrak{S}$-connected and hence contained in $\mathcal{C}_{\mathfrak{S}}$. □

Note that neither axiom (S0) nor (S2) is used here. The following result parallels the second part of Prop. 3.3 in [26]:

Lemma 3.9. *Given a symmetric separation $\mathfrak{S}$ on X satisfying (SR0), (SR1), and (SR2), the $\mathfrak{S}$-connected components of a set $Y \subseteq X$ are exactly the sets Y_j described in condition (SR3). Furthermore, $Y^\circ = \mathring{Y}$.*

Proof. By Lemma 3.3, $\mathfrak{S}$ satisfies (SR3). Property (SR3;i) immediately implies that the sets Y_j are $\mathfrak{S}$-connected. Consider an arbitrary subset $Y \subseteq X$, let $Y_j \subseteq Y$ and denote by Z_j the $\mathcal{C}_{\mathfrak{S}}$-connected component of Y that contains Y_j. By (SR3;ii) we have $(Y_j, Y \setminus Y_j) \in \mathfrak{S}$ and since $Z_j \in \mathcal{C}_{\mathfrak{S}}$ we have $Z_j \cap Y \setminus Y_j = \emptyset$, i.e. $Z_j \subseteq Y_j$. Since $Y_j \subseteq Z_j$ by construction, we conclude $Z_j = Y_j$, i.e., the Y_j are the connected components of Y as claimed.

We have $y \in Y^\circ$ if and only if y is not contained in a connected component of Y, i.e., if and only if there is a pair $(A, B) \in \mathfrak{S}$ such that $Y \subseteq A \cup B$ and $y \in A \cap B$. This is equivalent to $y \in \mathring{Y}$ by equ.(3.1). □

Lemma 3.10. *Let $\mathfrak{S}$ be a symmetric separation satisfying (SR0), (SR1), and (SR2). If (A, B) is $\mathcal{C}_{\mathfrak{S}}$-separating, then $(A, B) \in \mathfrak{S}$.*

Proof. Suppose (A, B) is $\mathcal{C}_{\mathfrak{S}}$-separating. Then either $A = \emptyset$ or $B = \emptyset$, in which case (A, B) is trivially $\mathfrak{S}$-separating, or both A and B are non-empty, in which case $A \cup B \notin \mathcal{C}_{\mathfrak{S}}$. Let Y_j, $j \in J$ denote components of $Y := A \cup B$ described in (SR3). By Lemma 3.9 they are connected components of Y in $\mathcal{C}_{\mathfrak{S}}$. Therefore $Y_j \cap A = \emptyset$ or $Y_j \cap B = \emptyset$, i.e., Y_j is contained either in A or in B. With $J_A := \{j|Y_j \subseteq A\}$ and

$J_B := \{j|Y_j \subseteq B\}$ we have $A = \bigcup_{j\in J_A} Y_j \cup (A\cap Y^\circ)$ and $B = \bigcup_{j\in J_B} Y_j \cup (B\cap Y^\circ)$, which by (SR3;ii) implies $(A, B) \in \mathfrak{S}$. □

Lemma 3.11. *If $Z \notin \mathcal{C}$ then Z is not $\mathfrak{S}_\mathcal{C}$-connected.*

Proof. Suppose $Y \notin \mathcal{C}$ and let U be a connected component of Y. Since Y is not connected, $U \subset Y$ and hence $V = Y \setminus U \neq \emptyset$. As a consequence of Lemma 3.6, every connected component of $Y = U \cup V$ is either contained in U or in V, hence $(U, V) \in \mathfrak{S}_\mathcal{C}$. Since $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ it follows that Y is not $\mathfrak{S}_\mathcal{C}$-connected. □

We can summarize these two lemmas as $\mathcal{C}_{\mathfrak{S}_\mathcal{C}} \subseteq \mathcal{C}$ and $\mathcal{S}_{\mathfrak{C}_\mathcal{S}} \subseteq \mathcal{S}$.

Galois Connection. For the reminder of this subsection, the pairs of the form $(A, \emptyset)$, which were included by axiom (S0) following [12, 13] to avoid having to exclude empty sets explicitly in much of the discussion become a bit of a nuisance. We therefore set $\widehat{\mathfrak{S}} := \mathfrak{S} \setminus \mathfrak{N}$, where $\mathfrak{N}$ denotes the set of all pairs $(A, \emptyset)$ and $(\emptyset, B)$. Obviously, there is a one-to-one correspondence between $\mathfrak{S}$ and $\widehat{\mathfrak{S}}$. Furthermore, we define

$$\mathfrak{D} := \{\{A; B\}|A \neq \emptyset, B \neq \emptyset\} \tag{3.2}$$

where $\{A; B\}$ denotes an unordered pair of subsets of X and we identify the two ordered pairs (A, B) and (B, A) with the single unordered pair $\{A; B\}$. With this slight abuse of notation, we have $\widehat{\mathfrak{S}} \subseteq \mathfrak{D}$ if and only if $\mathfrak{S}$ satisfies (S0) and (S1).

It is easy to check that both $\mathfrak{D}$ and 2^X form a complete lattice. Furthermore, if two symmetric separations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ give rise to the same collections of $\mathfrak{S}_i$-connected sets, their union $\mathfrak{S}_1 \cup \mathfrak{S}_2$ again generates the same collection of $\mathfrak{S}$-connected sets. Furthermore, axioms (S0), (S1), (S2), (SR0), (SR1), and (SR2) remain intact in $\bigcup_{\imath\in I} \mathfrak{S}_\imath$ and $\bigcap_{\imath\in I} \mathfrak{S}_\imath$ whenever $\mathfrak{S}_\imath$ have the same $\mathfrak{S}$-connected sets for all $\imath \in I$.

We introduce the binary relation $\between$ on $\mathfrak{D} \times 2^X$ by setting $\{A; B\} \between Z$ if and only if $Z \subseteq A \cup B$ implies $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$. Note that this same definition also makes perfect sense for ordered pairs (A, B). Finally, we introduce two maps Σ and Φ and follows:

$$\begin{aligned}
\Sigma :& 2^{2^X} \to 2^{\mathfrak{D}} : \mathcal{C} \mapsto \Sigma(\mathcal{C}) \\
&\{A; B\} \in \Sigma(\mathcal{C}) \Leftrightarrow \forall Z \in \mathcal{C},\ \{A; B\} \between Z \\
\Phi :& 2^{\mathfrak{D}} \to 2^{2^X} : \widehat{\mathfrak{S}} \mapsto \Phi(\widehat{\mathfrak{S}}) \\
&Z \in \Phi(\widehat{\mathfrak{S}}) \Leftrightarrow \forall \{A; B\} \in \widehat{\mathfrak{S}}, \{A; B\} \between Z
\end{aligned} \tag{3.3}$$

The maps Σ and Φ by construction form a Galois connection. Theorem 3.8 shows that $\Phi(\widehat{\mathfrak{S}})$ is a connectivity space, i.e., it satisfies (c0) and (c1). Theorem 3.2 shows that $\Sigma(\mathcal{C})$ is a symmetric separation satisfying (SR0), (SR1), and (SR2).

In terms of Σ and Φ, we can recast Lemmas 3.10 and 3.11 as $\Sigma(\Phi(\mathfrak{S})) \subseteq \mathfrak{S}$ and $\Phi(\Sigma(\mathcal{C})) \subseteq \mathcal{C}$. The theory of Galois connections establishes the reverse inclusions, see e.g. [52] or the discussion in [26].

We summarize this reasoning in the following

Theorem 3.12. *There is a one-to-one correspondence between connectivity spaces and symmetric separations satisfying axioms (SR0), (SR1), and (SR2) such that*

$$\begin{aligned} \mathfrak{S}_{\mathfrak{C}} &= \left\{(A,B) \subseteq 2^X \times 2^X | (A,B) \between Z \text{ for all } Z \in \mathfrak{C}\right\} \\ \mathfrak{C}_{\mathfrak{S}} &= \left\{Z \subseteq 2^X | (A,B) \between Z \text{ for all } (A,B) \in \mathfrak{S}\right\} \end{aligned} \tag{3.4}$$

Finally, we record two simple consequences of (SR3) and the fact that the sets Y_j are the connected components of $Y \subseteq X$.

Fact 3.13. *If $(X, \mathfrak{C})$ satisfies (c1) and $Z', Z'' \in \mathfrak{C}$, then either $Z' \cup Z'' \in \mathfrak{C}$ or $(Z', Z'') \in \mathfrak{S}_{\mathfrak{C}}$.*

Fact 3.14. *For every $A \subseteq X$ and $x \in A$ we have $(A[x], A \setminus A[x]) \in \mathfrak{S}$*

3.4. The (SR0) Axiom Revisited

The explicit use of $\mathring{Y}$ in axiom (SR0) lacks a certain conciseness, although it turned out to be particularly convenient to handle the properties of the "disconnected points" in the proofs above. We show here that it can be replaced by a more elegant axiom. We start with simple characterization of $\mathring{Y}$:

Fact 3.15. *Suppose $\mathfrak{S}$ satisfies (S0), (S1), (S2), and (SR0). Then $A \subseteq \mathring{Y}$ if and only if $A \subseteq Y$ and $(A, Y) \in \mathfrak{S}$.*

Proof. Suppose $A \subseteq \mathring{Y}$. From (S0) we know $(Y, \emptyset) \in \mathfrak{S}$ and (SR0) implies $(Y, \mathring{Y}) \in \mathfrak{S}$. By heredity we also have $(Y, A) \in \mathfrak{S}$ for all $A \subseteq \mathring{Y}$. Conversely, if $(Y, A) \in \mathfrak{S}$ for $A \subseteq Y$ then by definition $A \subseteq \mathring{Y}$. □

An immediate consequence of Fact 3.15 is

Fact 3.16. *Suppose $\mathfrak{S}$ satisfies (S0), (S1), (S2), and (SR0). Then*

(SR0-1) *$A_\iota \subseteq B$ and $(A_\iota, B) \in \mathfrak{S}$ for all $\iota \in I$ implies $(\bigcup_{\iota \in I} A_\iota, B) \in \mathfrak{S}$.*

Lemma 3.17. *Suppose $\mathfrak{S}$ satisfies (S0), (S1), and (S2). Then (SR0) is equivalent to (SR0-1) and*

(SR0-2) *$(A \cup B \cup C, A) \in \mathfrak{S}$ and $(B, C) \in \mathfrak{S}$ if and only if $(A \cup B, A \cup C) \in \mathfrak{S}$.*

Proof. Suppose (SR0) holds. We first observe that $Y \setminus \mathring{Y} \subseteq B \cup C \subseteq Y$ implies that here is a set $A \subseteq \mathring{Y}$ such that $A \cup B \cup C = Y$. Fact 3.15 then implies that $A \subseteq \mathring{Y}$ and $(A \cup B \cup C, A) \in \mathfrak{S}$ are equivalent. By (SR0) $(\mathring{Y} \cup B, \mathring{Y} \cup C) \in \mathfrak{S}$, and by (S2) also $(A \cup B, A \cup C) \in \mathfrak{S}$. Thus $(A \cup B \cup C, A) \in \mathfrak{S}$ and $(B, C) \in \mathfrak{S}$ implies $(A \cup B, A \cup C) \in \mathfrak{S}$. Conversely, $(A \cup B, A \cup C) \in \mathfrak{S}$ implies $(B, C) \in \mathfrak{S}$ and $A \subseteq \mathring{Y}$. The latter inclusion implies $(A \cup B \cup C, A) \in \mathfrak{S}$ by virtue of Fact 3.15. Thus (SR0-2) holds.

Now suppose (SR0-1) and (SR0-2) hold. Consider three sets A, B, and C and set $Y := A \cup B \cup C$. Hence we have $Y \setminus A \subseteq B \cup C \subseteq Y$. By (SR0-2), $(B, C) \in \mathfrak{S}$ implies $(A \cup B, A \cup C) \in \mathfrak{S}$ for all A satisfying $(A \cup B \cup C, A) \in \mathfrak{S}$. Every such set

A is, by definition, contained in $\mathring{Y}$.
Now consider all sets A_ι for which there is a separation of the form $(A_\iota \cup B_\iota, A_\iota \cup C_\iota) \in \mathfrak{S}$ with $A_\iota \cup B_\iota \cup C_\iota = Y$. The definition of $\mathring{Y}$ amounts to $\mathring{Y} = \bigcup_\iota A_\iota$. Property (SR0-1) thus implies $(Y, \mathring{Y}) = (\mathring{Y} \cup B \cup C, \mathring{Y}) \in \mathfrak{S}$. Invoking (SR0-2) again, we arrive at $(B \cup \mathring{Y}, C \cup \mathring{Y}) \in \mathfrak{S}$, i.e., (SR0) is satisfied. □

Lemma 3.18. *Suppose $\mathfrak{S}$ satisfies (S0), (S1), (S2), (SR1), and (SR2). Then (SR0) is equivalent to*

(SR0-0) $(A \cup B, A \cup C) \in \mathfrak{S}$ *implies* $(A \cup B \cup C, A) \in \mathfrak{S}$.

Proof. We first observe that (SR0-1) is a special case of axiom (SR2) and thus can be dropped entirely. Furthermore, if $(A \cup B \cup C, A) \in \mathfrak{S}$ and $(B, C) \in \mathfrak{S}$ then by heredity (S2) we also have $(B \cup C, A) \in \mathfrak{S}$. Symmetry (S1) and property (SR1) imply $(C, B \cup A) \in \mathfrak{S}$. Applying (SR1) again to $(B \cup A, C) \in \mathfrak{S}$ and $((B \cup A) \cup C, A) \in \mathfrak{S}$ yields $(B \cup A, C \cup A) \in \mathfrak{S}$, i.e., the "only if" part of (SR0-2) is already a consequence of (SR1), hence we can replace (SR0) by the "if"-part of (SR0-2). □

3.5. Subspaces

A subset $Y \subseteq X$ inherits the connectivity structure $\mathcal{C}$ of X by means of $\mathcal{C}_Y = \{A \in \mathcal{C} | A \subseteq Y\}$. Analogously, the separation relation $\mathfrak{S}$ is inherited by means of $(A, B) \in \mathfrak{S}_Y$ if and only if $(A, B) \in \mathfrak{S}$ and $A, B \subseteq Y$.

4. Isotonic Closure Spaces

4.1. Kuratowski's Axioms and the Wallace Function

The connection between separations and isotonic closure spaces has been investigated already in the mid 20th century [12, 14, 15] in the context of Wallace separations and even more restrictive variants of proximity spaces [53, 50, 54]. In 2005, Harris [55] considered the general case of symmetric separations satisfying only (S1) and (S2).

Definition 4.1. [12, 14] Let $\mathfrak{S} \subseteq 2^X \times 2^X$ be an arbitrary relation. Its *Wallace function* $w : 2^X \to 2^X$ is defined as

$$w(A) = \bigcap \{B \subseteq X | (A, X \setminus B) \in \mathfrak{S}\}$$

Fact 4.2. [13] *If $\mathfrak{S}$ is a symmetric separation, i.e., if axioms (S1) and (S2) are satisfied, then the Wallace function can be written as*

$$w(A) = \{x \in X | (\{x\}, A) \notin \mathfrak{S}\} \tag{4.1}$$

Topological spaces and their generalization can be characterized by Kuratowski's axioms [56] for a closure function $c : 2^X \to 2^X$:

(K0) $c(\emptyset) = \emptyset$. (grounded)
(K1) $A' \subseteq A$ implies $c(A') \subseteq c(A)$ for all $A \subseteq X$. (isotone)
(K2) $A \subseteq c(A)$ for all $A \subseteq X$. (enlarging)

(K3) $c(A\cup B)\subseteq c(A)\cup c(B)$ for all $A,B\subseteq X$. (subadditive)
(K4) $c(c(A))=c(A)$ for all $A\subseteq X$. (idempotent)

A pair (X,c) satisfying (K0) and (K1) is a often called a *closure space*. In this setting, (K3) can be replaced by $c(A\cup B)=c(A)\cup c(B)$. (X,c) is a *neighborhood space* if in addition it is enlarging. A *pretopology* is also subadditive, and a *topology* satisfies all five axioms. If (X,c) satisfies (K0), (K1), (K2), and (K4), then it is a general convexity.

It follows directly from Definition 4.1 that w is isotone if $\mathfrak{S}$ is hereditary (S2). Furthermore (S0) implies $(\emptyset,X)\in\mathfrak{S}$ and hence $w(\emptyset)=\emptyset$, i.e., (K0).

A closure space is point-wise symmetric if it satisfies

(R0) $x\in c(\{y\})$ implies $y\in c(\{x\})$ for all $x,y\in X$.

Since by definition $w(\{x\})=\{y\in X|(\{x\},\{y\})\notin\mathfrak{S}\}$, we have $y\in w(\{x\})$ if and only if $(\{x\},\{y\})\notin\mathfrak{S}$, i.e., if and only if $(\{y\},\{x\})\notin\mathfrak{S}$, i.e., iff $x\in w(\{y\})$. Thus w satisfies the symmetry axiom (R0).

The pair (X,w) therefore is a point-wise symmetric closure space.

We note for later reference that, as a direct consequence the definition of the Wallace function, the set of non-connected points can be represented as

$$\mathring{Y}=\{y\in Y|y\notin w(Y)\}=Y\setminus w(Y) \tag{4.2}$$

4.2. Hausdorff-Lennes Separations

Definition 4.3. Let (X,c) be a closure space. Then we call

$$\mathfrak{S}_{HL}=\left\{(A,B)\in 2^X\times 2^X\middle|(A\cap c(B))\cup(c(A)\cap B)=\emptyset\right\} \tag{4.3}$$

the *Hausdorff-Lennes* separation of (X,c)

It follows immediately from the definition that $\mathfrak{S}_{HL}$ is a symmetric separation, i.e., that (S0), (S1), and (S2) hold. The Hausdorff-Lennes condition, equ.(4.3), gives the textbook definition of connectedness w.r.t. a topological space.

The following property plays a key role in the theory of separations and their associated closure spaces:

(SX) If $(\{x\},B)\in\mathfrak{S}$ for all $x\in A$ and $(A,\{y\})\in\mathfrak{S}$ for all $y\in B$ then $(A,B)\in\mathfrak{S}$.

The following result is a variant of (3.3.) in [12], Theorem 3.2 of [14], see also [50] and [55, Thm.3]:

Theorem 4.4. *If $\mathfrak{S}$ satisfies (S1) and (S2) and (SX) then $(A,B)\in\mathfrak{S}$ if and only if $A\cap w(B)=\emptyset$ and $w(A)\cap B=\emptyset$, i.e., $\mathfrak{S}$ is the Hausdorff-Lennes separation of the point-wise symmetric isotone closure space (X,w).*

The properties of point-symmetric closure spaces are therefore characterized by their Hausdorff-Lennes separations.

Fact 4.5. *If $\mathfrak{S}$ satisfies (S2) and (SX), i.e., if the corresponding closure function w is point-wise symmetric (R0), then $\mathfrak{S}$ is "grounded", i.e., satisfies (S0), if and only if (X,w) is grounded, i.e., if (K0) holds.*

Proof. In [55] the equivalence of (K0) with the (in general weaker) axiom "$(\{x\},\emptyset) \in \mathfrak{S}$ for all $x \in X$" is shown. This property is equivalent to (S0) whenever $\mathfrak{S}$ is symmetric (S2) and satisfies (SX). □

4.3. Topological Connectivity Spaces

Definition 4.6. Let (X, c) be a closure space. We say that $Z \subseteq X$ is *topologically connected* if it is $\mathfrak{S}_{HL}$-connected.

The discussion in the previous subsection established that closure spaces are equivalent to symmetric separations satisfying (SX). This does not, however, imply that they are equivalent to connectivity spaces. The following examples show that (SX) is independent of the axioms (SR1) and (SR2). Thus there are in general distinct closure spaces that give rise to the same connectivity space.

In the following two examples we use the shorthand notation $ab|c$ to mean $(\{a, b\}, \{c\}) \in \mathfrak{S}$ here.

Example. Set $X = \{a, b, c\}$ and suppose $\mathfrak{S}$ consists of trivial pairs stipulated by (S0) and the non-trivial separating pairs $a|b$, $a|c$, $b|c$, and $ab|c$ and their symmetric counterparts. It is straightforward to check that (S0), (S1), and (S2) holds. From $a|b$ and $ab|c$ (SR1) would imply $b|ac$, a contradiction.

Example. Conversely, set $X = \{a, a', b, b'\}$ and suppose $\mathfrak{S}$ contains the non-trivial separating pairs $a|bb'$, $a'|bb'$, $b|aa'$, $b'|aa'$ as well as the pairs $b|ab'$, $b'|ab$, $b|a'b'$, $b'|a'b$, $a|ba'$, $a'|ba$, $a|ab'$, and $a'|ab'$ implied by (SR1) as well as their symmetric counterparts and the subsets $a|b$, etc., implied heredity. Here (S0), (S1), (S2), and (SR1) hold by construction. Axiom (SX), however, would imply that $\mathfrak{S}$ also contains $aa'|bb'$ from $a|bb'$, $a'|bb'$, $b|aa'$ and $b'|aa'$, a contradiction.

Corollary 4.7. *A symmetric separation $\mathfrak{S}$ is generated by the topologically connected sets of a point-symmetric closure space if and only if $\mathfrak{S}$ satisfies (SX), (SR0), (SR1), and (SR2).*

Proof. By theorem 3.12 (SR0), (SR1), and (SR2) are necessary and sufficient for $\mathfrak{S}$ to be the separation of a connectivity space. By theorem 4.4, (SX) is necessary and sufficient for $\mathfrak{S}$ to be the Haussdorf-Lennes separation of a closure space. □

For symmetric separations satisfying (SX) it is straightforward to translate axioms (SR0), (SR1), (SR2) to Wallace functions making use of the equivalence

$$(A, B) \in \mathfrak{S} \quad \Leftrightarrow A \cap w(B) = \emptyset \wedge B \cap w(A) = \emptyset\,. \tag{4.4}$$

In the presence of (SX) we can rephrase (SR0-2), (SR1), and (SR2) as additional axioms for closure spaces:

(KR0) $(A\cup B\cup C)\cap c(A) = \emptyset$, $c(A\cup B\cup C)\cap A = \emptyset$, $B\cap c(C) = \emptyset$ and $c(B)\cap C = \emptyset$ holds and only if $(A \cup B) \cap c(A \cup C) = \emptyset$ and $c(A \cup B) \cap (A \cup C) = \emptyset$.

(KR1) $c(A) \cap B = c(B) \cap A = c(C) \cap (A \cup B) = C \cap c(A \cup B) = \emptyset$ implies $A \cap c(B \cup C) = c(A) \cap (B \cup C) = \emptyset$.

(KR2) If $A_i \cup B_i = Y$ for all $i \in I$ and $c(A_i) \cap B_i = A_i \cap c(B_i) = \emptyset$ then $\bigcap_{i\in I} A_i \cap c\left(\bigcup_{i\in I} B_i\right) = \emptyset$ and $c\left(\bigcap_{i\in I} A_i\right) \cap \bigcup_{i\in I} B_i = \emptyset$.

To our knowledge these properties have not been studied in the context of closure spaces so far.

Axiom (SX) does not hold for the $\mathcal{C}$-separations of arbitrary connectivity spaces. It is of interest therefore, to better understand it in terms of connected sets. It will be convenient in many of the subsequent arguments to use the following negative re-statement of Definition 3.1:

Fact 4.8. *Let $\mathfrak{S}$ be a a symmetric separation satisfying (SR0), (SR1), and (SR2) and let $\mathcal{C}$ be the equivalent connectivity space. Then $(A,B) \notin \mathfrak{S}$ if and only there is $Z \in \mathcal{C}$ such that $Z \cap A \neq \emptyset$ and $Z \cap B \neq \emptyset$.*

An immediate consequence of Fact 4.8 and equ.(4.1) is

Lemma 4.9. *Suppose $\mathfrak{S}$ is a symmetric separation satisfying (SR0), (SR1), and (SR2) and let $\mathcal{C}$ be the equivalent connectivity space. Then its Wallace function is of the form $x \in w(A)$ if and only if there is a connected set $Z \subseteq A \cup \{x\}$ such that $Z \cap A \neq \emptyset$ and $x \in Z$.*

Thus the elements of $w(A)$ are exactly the *touching points* of [19], see also [14, Thm.4.9].

In isotone closure spaces some of the theorems regarding connected sets that are well known from topological spaces still hold for the topological (Hausdorff-Lennes) connectivity. In particular, if A is topologically connected and $A \subseteq c(A)$, then $c(A)$ is topologically connected [57]. Following Thm.1.4. of [7] we also conclude that B is topologically connected whenver $A \subseteq B \subseteq c(A)$ and A is topologically connected.

We conclude this section by expressing axiom (SX) in terms of connected sets. Let us say that A and B form a proper bipartition of Z, in symbols $Z = A \dot{\cup} B$, whenever $Z = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap B = \emptyset$. Then we can state

(cX) For all $Z \in \mathcal{C}$ and all $A \dot{\cup} B = Z$ there is $x \in A$ and $\emptyset \neq B' \subseteq B$ such that $\{x\} \cup B' \in \mathcal{C}$ or $y \in B$ and $\emptyset \neq A' \subseteq A$ such that $A' \cup \{y\} \in \mathcal{C}$.

Lemma 4.10. *Suppose $\mathfrak{S}$ is a symmetric separation satisfying (SR0), (SR1), and (SR2) and let $\mathcal{C}$ be the equivalent connectivity space. Then $\mathfrak{S}$ fulfils (SX) if and only if $\mathcal{C}$ satisfies (cX).*

Proof. It will be convenient to express (SX) in its negated form: If $(A,B) \notin \mathfrak{S}$ then there is $x \in A$ such that $(\{x\},B) \notin \mathfrak{S}$ or $y \in B$ such that $(A,\{y\}) \notin \mathfrak{S}$. We assume (S0), (S1), (S2), and (c0), (c1), respectively.

Now suppose (SX) holds and consider $Z \in \mathcal{C}$ and a pair of non-empty subsets A and B of Z such that $A \cup B = Z$. In particular, we may assume that A and B are disjoint. By assumption, $(A,B) \notin \mathfrak{S}$ and hence, by (SX), there is $x \in A$ such that $(\{x\},B) \notin \mathfrak{S}$ or $y \in B$ such that $(A,\{y\}) \notin \mathfrak{S}$. By Fact 4.8 there is $Z' \in \mathcal{C}$ such that $Z' = \{x\} \cup B'$ with $B' \subseteq B$, or there is $Z'' \in \mathcal{C}$ such that $Z'' = \{y\} \cup A'$ with $A' \subseteq A$, i.e., (cX) holds.

Conversely, suppose (cX) holds. Consider a pair of sets $C, D \in 2^X$ such that $(C,D) \notin \mathfrak{S}$. Then by Fact 4.8 there is a $Z \in \mathcal{C}$, $Z \subseteq C \cup D$ with $A := Z \cap C \neq \emptyset$

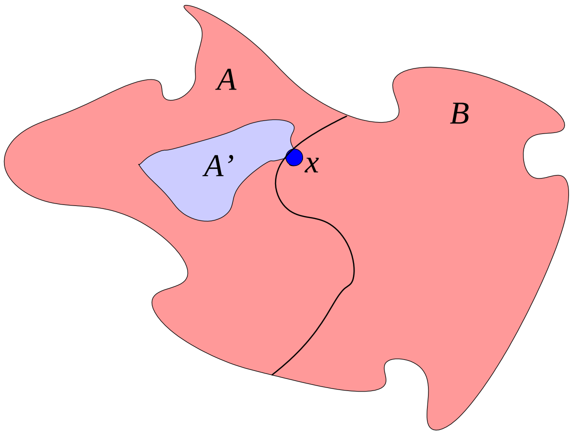

FIGURE 5. Illustration of axiom (cX). Every connected set Z can be bipartitioned in two arbitrary subsets that can be labeled A and B in such a way that there is a non-empty subset $A' \subseteq A$ and a single point $x \in B$ such that $A' \cup \{x\}$ is connected.

and $B := Z \cap D \neq \emptyset$. By (cX) there is $x \in A$ and $\emptyset \neq B' \subseteq B \subseteq D$ with $Z' := \{x\} \cup B' \in \mathfrak{C}$ or $y \in B$ and $\emptyset \neq A' \subseteq A \subseteq C$ with $Z'' := \{y\} \cup A' \in \mathfrak{C}$. If (SX) does not hold, then by Fact 4.8 $(\{x\}, B') \notin \mathfrak{S}$ or $(\{y\}, A') \notin \mathfrak{S}$, a contradiction. $\square$

Corollary 4.11. *A connectivity $\mathfrak{C}$ of X consists of the HL-connected sets of a closure space (X, c) if and only if it satisfies (c0), (c1), and (cX).*

Axiom (cX) implies that every connected set with at least 4 points contains a strictly smaller connected set comprising at least 2 points. In [19] a strictly weaker condition has been investigated:

(c3) If $Y, Z, Y \cup Z \in \mathfrak{C}$ then there is $z \in Y \cup Z$ such that $Y \cup \{z\} \in \mathfrak{C}$ and $Z \cup \{z\} \in \mathfrak{C}$.

Lemma 4.12. *If $\mathfrak{C}$ satisfies (c0) and (c1), then (cX) implies (c3).*

Proof. Consider $Z, Z', Z'' \in \mathfrak{C}$ such that $Z' \cup Z'' = Z$. By (cX) there is w.l.o.g. $x \in Z'$ and $Y \subseteq Z''$ such that $\{x\} \cup Y \in \mathfrak{C}$. By (c1) this implies $\{x\} \cup Z'' \in \mathfrak{C}$ and $\{x\} \cup Z' = Z' \in \mathfrak{C}$ by definition. $\square$

We note that (c3) does not imply (cX) because there is, in general no guarantee that a particular bipartition of a connected set $Z \in \mathfrak{C}$ into two non-empty sets $A \cup B = Z$ is such that A or B contains a connected set. Therefore, there are connectivity spaces satisfying (c3) that do not derive from closure spaces.

5. Integral Connectivity Spaces

5.1. Integral Connectivity Spaces

In most settings it is natural to require that singletons (individual points) are connected. This amounts to the axiom

(c2) $\{x\} \in \mathcal{C}$ for all $x \in X$

Spaces $(X, \mathcal{C})$ satisfying (c0), (c1), and (c2) are called *connections* in [21] and *integral connectivity spaces* in [20], and *c-spaces* in [19]. We will use the term integral connectivity space here.

In terms of the connectivity openings, (c2) is equivalently expressed as

(o5) $\{x\}[x] = \{x\}$.

We remark that (o5) can be expressed equivalently as "$x \in A[x]$ for all $x \in A$ and all $A \in 2^X$." Thus the sets $\{A[x]|x \in A\}$ form a partition of A. In the presence of (o5) one can replace (o4) equivalently by the weaker condition

(o4') $A[x] \cap A[y] = \emptyset$ or $A[x] = A[y]$.

Details can be found in [41, Lemma 19].

It is natural therefore, to complete a basis $\mathcal{B}$ by adding all singletons to $\mu'(\mathcal{B})$, i.e., $\mu(\mathcal{B}) := \mu'(\mathcal{B}) \cup \{\{x\}|x \in X\}$ [41, Prop.24]. It is obvious that $\mu(\mathcal{B})$ is the smallest collection of subsets of X satisfying (c0), (c1), and (c2).

In particular, $\mathring{Y} = \emptyset$ if (c2) holds. As an immediate consequence, the corresponding separation $\mathfrak{S}$ satisfies

(S3) $(X, Y) \in \mathfrak{S}$ implies $X \cap Y = \emptyset$.

Ronse showed in [26] that there is a bijection between integral connectivity spaces and symmetric separations satisfying (S3), (SR1), and (SR2). We remark that one recovers Ronse's phrasing of (SR2) by making explicit use of the disjunctive axiom (S3) to replace $B_i \to Y \setminus A_i$.

5.2. Wallace Separations

Symmetric, grounded, hereditary, and disjunctive relations $\mathfrak{S}$ on 2^X, i.e., the ones that satisfy (S0), (S1), (S2), and (S3) were considered already by Wallace in the 1940s [12] and were later investigated under the name *Wallace separations* [14, 15]. Most of the results outlined in the previous sections for the non-disjunctive setting were obtained for the disjunctive case in this classical literature. We call $(X, \mathfrak{S})$ a *Wallace separation space* if $\mathfrak{S}$ satisfies (S0), (S1), (S2), and (S3).

Lemma 5.1. *Suppose $\mathcal{C}$ satisfies (c0) and (c1) and $\mathfrak{S}$ is the corresponding symmetric separation (which satisfies (SR1) and (SR2)). Then $\mathfrak{S}$ is a Wallace separation, i.e., $\mathfrak{S}$ satisfies (S3), if and only if $\mathcal{C}$ satisfies (c2).*

Proof. Suppose $\mathfrak{S}$ is disjunctive (S3). Then, by definition $\{x\}$ is $\mathfrak{S}$-connected for all $x \in X$ since for all $(A, B) \in \mathfrak{S}$ with $x \in A \cup B$ and $A \cap B = \emptyset$, either $x \notin A$ or $x \notin B$. Conversely, suppose $\{x\} \in \mathcal{C}$ for all $x \in X$. Then $(A, B) \in \mathfrak{S}$ if $x \in A \cup B$ implies $x \notin A$ or $x \notin B$. Thus $A \cap B = \emptyset$. □

Fact 5.2. [55, 13] *If $\mathfrak{S}$ satisfies (S1), and (SX), i.e., the corresponding Wallace function w is pointwise symmetric (R0), then $\mathfrak{S}$ is disjunctive (S3) if and only if (X, w) is enlarging (K2).*

The maximal separations satisfying (S0), (S1), (S2), (S3), and (SX) thus correspond exactly to the symmetric *neighborhood spaces*, i.e., the closure spaces satisfying (K0), (R0), (K1), and (K2).

Again, there are in general multiple non-isomorphic closure spaces that give rise to the same integral connectivity space since the axioms (S0), (S1), (S2), (S3), and (SX) do not imply (SR1) and (SR2).

Fact 5.3. [57, Thm. 5.2] *An isotone closure space (X, w) is connected if and only if it is not the disjoint union of two closed sets, where a set A is closed if $A = c(A)$.*

5.3. Connectedness of Generalized Topologies sensu Császár

A "generalized topological space" (GTS) in the sense of Császár [6] consists of a set system $\mathcal{O} \subseteq 2^X$ such that (i) $\emptyset \in \mathcal{O}$ and (ii) $O_\iota \in \mathcal{O}$ for $\iota \in I$ implies $\bigcup_{\iota\in I} O_\iota \in \mathcal{O}$. The elements of $\mathcal{O}$ are regarded as generalized open sets, their complements are closed sets. For each $A \subseteq X$ denote by $j(A)$ the union of all open sets contained in A and by $k(A)$ the intersection of all closed sets containing A. By construction, $k : 2^X \to 2^X$ is a generalized closure operator satisfying (K1), (K2), and (K4). The associated interior operator $j(A)$ satisfies $j(A) = X \setminus k(X \setminus A)$. Property (K0) is equivalent to requiring that, in addition, $X \in \mathcal{O}$.

A set $A \subseteq X$ is
semi-open, $A \in \mathcal{O}^\sigma$, iff $A \subseteq kj(A)$;
pre-open, $A \in \mathcal{O}^\pi$, iff $A \subseteq jk(A)$;
α-open, $A \in \mathcal{O}^\alpha$, iff $A \subseteq jkj(A)$;
β-open, iff $A \in \mathcal{O}^\beta$, $A \subseteq kjk(A)$ [58].
Note that if A is open, it is also semi-open, pre-open, α-open, and β-open. Since $kjkj = kj$ and $jkjk = jk$ [6] we see that jk, kj, jkj, kjk are idempotent (K4). A GTS is said to be *χ-connected*, for $\chi \in \{\sigma, \pi, \alpha, \beta\}$, if there are no two non-empty disjoint subsets $U, V \in \mathcal{O}^\chi$ such that $U \cup V = X$, see e.g. [7, 8, 9] for a systematic analysis. This definition generalizes Fact 5.3. Properties of the Hausdorff-Lennes separation w.r.t. χ are studied e.g. in [59].

In the most general setting, [7] considers an isotonic function $\gamma : 2^X \to 2^X$. Then $\mathcal{O}^\gamma = \{A | A \subseteq \gamma(A)\}$ is a GTS. Call a set γ-closed if $X \setminus A \in \mathcal{O}^\gamma$ and let k_γ be the corresponding generalized closure operator, which again satisfies (K0), (K1), (K2), and (K4). Two sets U and V are called γ-separated if they are Hausdorff-Lennes separated w.r.t. k_γ, i.e., $k_\gamma(U) \cap V = U \cap k_\gamma(V) = \emptyset$. Clearly, γ-separatedness defines a Wallace separation with corresponding Wallace function k_γ. The γ-connected sets of [7] are exactly the connected sets w.r.t. to this separation. This includes, in particular, also the specific GTS $\mathcal{O}^\chi$ mentioned in the previous paragraph.

We note, finally, that [7] also shows directly that the γ-connected sets satisfy (c1) and that γ-connected components are well-defined. Furthermore, the γ-connected components of a γ-closed set $Q = k_\gamma(Q)$ are also γ-closed.

6. The Additivity Axiom

A key role in the theory of separations and proximities is played by the following axiom of additivity:

(S4) $(A, B \cup C) \in \mathfrak{S}$ whenever $(A, B) \in \mathfrak{S}$ and $(A, C) \in \mathfrak{S}$

It was introduced already by Wallace [12], who called separation spaces $(X, \mathfrak{S})$ that satisfy (S0) through (S4) "s-spaces". They are considered under the name *basic proximity spaces* e.g. in [60] and as "Č-proximities" in [61].

A key consequence of (S4) is the classical "decomposition theorem" for connected topological spaces [12, 62]:

Theorem 6.1. *Let $(X, \mathcal{C})$ be a connectivity space, and let $\mathfrak{S} = \mathfrak{S}_{\mathcal{C}}$ be its corresponding symmetric separation. If $\mathfrak{S}$ satisfies (S4) then the following condition holds:*

(D) *$C, Z \in \mathcal{C}$, $C \subseteq Z$, $Z \setminus C := M \cup N$ and $(M, N) \in \mathfrak{S}$ implies $C \cup M \in \mathcal{C}$ and $C \cup N \in \mathcal{C}$.*

Proof. We follow [12, Thm.4.5.ii]. Suppose $C \cup M \notin \mathcal{C}$, i.e., there is $(A, B) \in \mathfrak{S}$ with $A \neq \emptyset$, $B \neq \emptyset$, and $A \cup B = C \cup M$ and thus $C \subseteq A \cup B$. Thus $C \in \mathcal{C}$, implies $A \cap C = \emptyset$ or $B \cap C = \emptyset$. W.l.o.g., we assume $A \cap C = \emptyset$, thus $A \subseteq M$, and by (S2) $(A, N) \in \mathfrak{S}$. From this and $(A, B) \in \mathfrak{S}$, we conclude using axiom (S4) that $(A, B \cup N) \in \mathfrak{S}$. But $Z = C \cup M \cup N = A \cup B \cup N$ is connected, thus either A or $B \cup N$, and hence B, must be empty, a contradiction. Thus $C \cup M \in \mathcal{C}$. Connectedness of $C \cup N$ is shown analogously. □

A variant of the Decomposition Theorem (D) can be stated in terms of connected sets only.

(c4a) Suppose $C, Z \in \mathcal{C}$, $C \subseteq Z$ and suppose W is a connected component of $Z \setminus C$. Then $C \cup W \in \mathcal{C}$.

A simple consequence of (c4a) is

Lemma 6.2. *Let $(X, \mathcal{C})$ be an integral connectivity space satisfying (c4a). Suppose $Z \in \mathcal{C}$, consider three pairwise disjoint non-empty $\mathcal{C}$-connected subsets $Z_i \subseteq Z$, $i = \{1, 2, 3\}$, and denote by $Z_{i|k}$ the connected component of $Z \setminus Z_k$ that contains Z_i for $i \neq k \in \{1, 2, 3\}$. Then at least two of the three conditions $Z_1 \subseteq Z_{2|3}$, $Z_2 \subseteq Z_{3|1}$, and $Z_3 \subseteq Z_{1|2}$ are satisfied.*

Proof. We distinguish two cases: (i) Z_2 and Z_3 are in distinct connected components W_2 and W_3 of $Z \setminus Z_1$. By (c4a) both $Z_1 \cup W_2 \in \mathcal{C}$ and $Z_1 \cup W_3 \in \mathcal{C}$ and hence $Z_1 \subseteq Z_{2|3}$ and $Z_3 \subseteq Z_{1|2}$. (ii) Z_2 and Z_2 are in the same connected component W of $Z \setminus Z_1$, i.e., $Z_2 \subseteq Z_{3|1}$. Consider $Z \setminus Z_2$. Now either Z_1 and Z_3 are in the same connected component – in which case $Z_3 \subseteq Z_{1|2}$ – or there are two distinct connected components of $Z \setminus Z_2$, say W_1' containing Z_1 and W_3' containing Z_3. The same argument as in (i) yields $Z_1 \subseteq Z_{2|3}$. □

In [19] a similar, but weaker, condition was proposed:

(c4b) Suppose $C, Z \in \mathcal{C}$, $C \subseteq Z$ and suppose W is a connected component of $Z \setminus C$. Then $Z \setminus W \in \mathcal{C}$.

Fact 6.3. *In an integral connectivity space (c4a) implies (c4b).*

Proof. Since (c2) holds, the set $Z \setminus C$ is partitioned into connected components W_ι, $\iota \in I$. By (c4a), $C \cup W_\iota \in \mathcal{C}$. Fixing a particular W_κ, $\kappa \in I$, (c1) implies $C \cup \bigcup_{\iota \in I \setminus \{\kappa\}} W_\iota = Z \setminus W_\kappa \in \mathfrak{C}$. □

Note that this implication is not true if (c2) does not hold.

Lemma 6.4. *Suppose $\mathcal{C}$ is an integral connectivity space. Then property (D) is equivalent to (c4a).*

Proof. Suppose (D) holds. Let $C, Z \in \mathcal{C}$, $C \subseteq Z$, and W a connected component of $Z \setminus C$. Define $U = (Z \setminus C) \setminus W$, i.e., $Z \setminus C = U \cup W$. By Fact 3.14 $(U, W) \in \mathfrak{S}$. By (D) we have $U \cup C = Z \setminus W \in \mathcal{C}$ and $W \cup C \in \mathcal{C}$.

Conversely, suppose (c4a) holds. Denote by W_ι, $\iota \in I$, the connected components of $Z \setminus C$. As above set $U_\iota = (Z \setminus C) \setminus W_\iota$, thus $(U_\iota, W_\iota) \in \mathfrak{S}$ for all $\iota \in I$. Since $(X, \mathcal{C})$ is integral, $\mathring{Y} = \emptyset$ and (SR3) reduces to its first part (SR3:i). It implies that a bipartition $M \dot{\cup} N$ of $Z \setminus C$ into $\mathfrak{S}$-separated sets, i.e., $(M, N) \in \mathfrak{S}$, must be of the form $M = \bigcup_{\iota \in J} W_\iota$ and $N = \bigcup_{\iota \in I \setminus J} W_\iota$ for any choice of I and J. By (c4a) $W_\iota \cup C \in \mathcal{C}$. Property (c1) now implies $M \cup C = \bigcup_{\iota \in J} (W_\iota \cup C)$ and $N \cup C = \bigcup_{\iota \in I \setminus J} (W_\iota \cup C)$ are connected, i.e., (D) holds. □

The following property was used in [19] as an alternative additivity axiom:

(c4b') Let $A, B, Z_\imath \in \mathcal{C}$ for all $\imath \in I$ and suppose $A \cup B \cup \bigcup\limits_{\imath \in I} Z_\imath \in \mathcal{C}$. Then there is $J \subseteq I$ such that $A \cup \bigcup\limits_{\jmath \in J} Z_\jmath \in \mathcal{C}$ and $\bigcup\limits_{\jmath \in I \setminus J} Z_\jmath \cup B \in \mathcal{C}$.

Note that we may assume, as in [19], that the sets A, B, and $Z_\imath$, $\imath \in I$ are pairwise disjoint since overlapping sets can just as well be unified if this is not the case initially.

Lemma 6.5. *Suppose $(X, \mathcal{C})$ satisfies (c0), (c1) and (c2). Then (c4b') and (c4b) are equivalent.*

Proof. Suppose (c4b') holds and let $Z_\imath \in \mathcal{C}$, $\imath \in I$ be the connected components of $Z \setminus C$ except for a particular connected component W. By (c2) we have $W \cup \bigcup_{\imath \in I} Z_\imath = Z \setminus C$. By (c4b') there is $J \subseteq I$ such that $C \cup \bigcup_{\jmath \in J} Z_\jmath \in \mathcal{C}$ and $\bigcup_{\jmath \in I \setminus J} Z_\jmath \cup W \in \mathcal{C}$. Since W is a connected component of $Z \setminus C$, no larger subset of $Z \setminus C$ containing W is connected, thus $J = I$, i.e., $C \cup \bigcup_{\imath \in I} Z_\imath = C \cup (Z \setminus C) \setminus W = Z \setminus W \in \mathcal{C}$.

To see the converse implication, let $A, B, Z_\imath \in \mathcal{C}$, $\imath \in I$, assume $Z := A \cup \bigcup_{\imath \in I} Z_\imath \cup B \in \mathcal{C}$, and let W be the connected component of $Z \setminus A$ containing B, which is necessarily of the form $W = B \cup \bigcup_{\jmath \in J} Z_\jmath$ for some $J \subseteq I$. Then (c4b) implies that $Z \setminus W = A \cup \bigcup_{\jmath \in I \setminus J} \in \mathcal{C}$, i.e., (c4b') holds. □

Note that in the absence of (c2) we can only conclude that (c4b) implies (c4b').

Lemma 6.6. *Let $(X, \mathcal{C})$ be a finite integral connectivity space. Then (c4a) and (c4b) are equivalent.*

Proof. We already know that (c4a) implies (c4b) in general. Thus suppose (c4b) holds. We proceed by induction in the number n of connected components of $Z \setminus C$. For $n = 0$ we have $Z \setminus C = \emptyset$ and $C = Z$. For $n = 1$, $W = Z \setminus C$ by (c2) and $C \cup W = Z \in \mathcal{C}$ by (c4b).

Now suppose the result is true for up to $n \geq 1$ components and suppose $Z \setminus C$ have $n + 1$ connected components. Given a connected component W of $Z \setminus C$, let W' be another connected component and set $Z' = Z \setminus W'$; by (c4b), $Z' \in \mathcal{C}$. By Fact 2.5 the connected components of $Z' \setminus C = (Z \setminus W') \setminus C = (Z \setminus C) \setminus W'$ are exactly the connected components of $Z \setminus C$ other than W'. Therefore $Z' \setminus C$ has n connected components among which is W. The induction hypothesis thus implies $C \cup W \in \mathcal{C}$. □

In the infinite case this proof idea of course fails. We have not been able, however, to construct a connectivity space satisfying (c4b) but not (c4a).

An even weaker variant of (c4b) was discussed in [19]:

(c4") If $Z_i \in \mathcal{C}$ for $i \in I = \{1, ..., n\}$ and $\displaystyle\bigcup_{i=1}^{n} Z_i \in \mathcal{C}$, then for every nonempty $J \subset I$ there is $j \in J$ and $k \in I \setminus J$ such that $Z_j \cup Z_k \in \mathcal{C}$.

Since n is finite we can view $V := \{Z_i | 1 \leq i \leq n\}$ as the vertices of a graph $G := G(\{Z_i\})$ with edges $i \sim k$ whenever $Z_i \cup Z_k \in \mathcal{C}$. Condition (c4") means that there is an edge across every vertex cut. Thus (c4") is equivalent to graph-theoretic connectedness of G. In particular, for every pair of sets Z_i, Z_j there is a path $P = (i =: i_0, i_1, \ldots, i_\ell := j)$ in G such that $Z_h \cup Z_{h+1} \in \mathcal{C}$ and any connected subgraph of G with vertex set V' corresponds to a collection $\{Z_i | i \in V'\}$ with $\bigcup_{i \in V'} Z_i \in \mathcal{C}$. In the following we will occasionally make use of this graph-theoretic interpretation of (c4").

Fact 6.7. *Axiom (c4") is equivalent to the following statement* (c4"') *If $Z_i \in \mathcal{C}$ for $i = 1, \dots, n$ and $\bigcup_{i=1}^{n} Z_i \in \mathcal{C}$, then there is a permutation π with arbitrary choice of $\pi(1)$ such that $\bigcup_{j=1}^{k} Z_{\pi(j)} \in \mathcal{C}$ for all $1 \le k \le n$.*

Proof. Given $G(\{Z_i\})$, the order π can be constructed by a breadth-first-search on $G(\{Z_i\})$ starting with $\pi(1)$, which exists as an immediate consequence of the graph-theoretic connectedness of $G(\{Z_i\})$.

To see the converse, we proceed by induction. For $n = 1, 2$ property (c4") is trival. In the general step we know that $G(\{Z_i\}_{i<n})$ is connected. Choose $\pi(1) = n$. Then by assumption $Z_n \cup Z_{\pi(2)} \in \mathcal{C}$, i.e, $\{Z_n, Z_{\pi(2)}\}$ is an edge in $G(\{Z_i\})$, and hence $G(\{Z_i\})$ is connected. □

Both properties (c4") and (c4"') have been described as properties of connected sets e.g. in Kuratowski's book on point set topology [62].

Lemma 6.8. *(c4b') implies (c4").*

Proof. Let $Z_i \in \mathcal{C}$ for $i = 1, ..., n$ and $Z := \bigcup_{i=1}^{n} Z_i \in \mathcal{C}$. We proceed by induction. For $n = 2$, (c4") is satisfied trivially. For $n = 3$, (c4b') implies that there is at least one pair of constituent sets, say Z_1 and Z_2, such that $Z_1 \cup Z_2 \in \mathcal{C}$. Now set $A = Z_1$ and $B = Z_2$. Then (c4b') implies that $Z_1 \cup Z_3 \in \mathcal{C}$ or $Z_2 \cup Z_3 \in \mathcal{C}$, the graph G with three vertices in the proof of fact 6.7 is connected. Now suppose $n > 3$. By (c4b') there is a subdivision of Z into two connected sets both comprising fewer constituents Z_i. Subdividing each of these further we obtain at least one connected set comprising exactly two constituents, i.e., there is $j \neq k$ such that $Z' := Z_j \cup Z_k \in \mathcal{C}$. Thus we can write Z as a union of $n-1$ connected sets, and hence by assumption the corresponding graph G' on $n-1$ vertices is connected. The vertex representing Z' is connected to a non-empty set $L = \{l_1, \dots l_u\}$ of vertices representing constituents Z_l. Thus $Z_j \cup Z_k \cup Z_l \in \mathcal{C}$ for all $l \in L$ and hence $Z_l \cup Z_j \in \mathcal{C}$ or $Z_l \cup Z_k \in \mathcal{C}$. Thus there is a path between l' and l'' for all $l', l'' \in L$, and hence the n-vertex graph G is connected if and only if the $n-1$-vertex graph G' is connected. As in the proof of fact 6.7, connectedness of G implies that (c4") is satisfied. □

Note that in the absence of (c2) we have only the implications (c4a) $\Longrightarrow$ (c4b') $\Longrightarrow$ (c4").

We remark that the equivalence of (c4b), (c4b'), and (c4") is claimed in [19] alluding to Kuratowski's book [62], but without hinting at a proof. We suspect that property (c4"), and thus also (c4"'), is in general strictly weaker than (c4b').

For $n = 3$ we may rephrase (c4") in the following form.

Fact 6.9. *Suppose (c0), (c1), and (c4a) holds, $Z', Z'', Z''' \in \mathcal{C}$ and $Z' \cup Z'' \cup Z''' \in \mathcal{C}$. Then at least two of the unions $Z' \cup Z''$, $Z' \cup Z'''$, and $Z'' \cup Z'''$ are connected.*

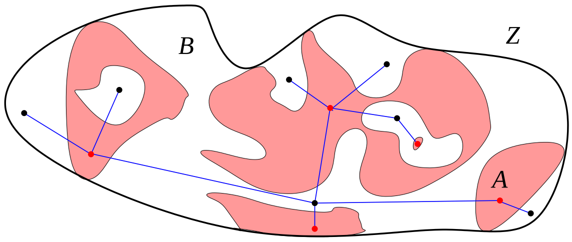

FIGURE 6. Illustration of axiom (c4). Any nontrivial, but otherwise arbitrary partition of a connected set Z into two disjoint subsets A (red) and B (white) has the property that for every connected component A' of A there is at least one connected component B' of B so that the union $A' \cup B'$ is again connected. All such pairs are indicated here by edges.

This can also be seen as a special case of Lemma 6.2. Note that Fact 6.9 does not imply (c4").

We next introduce a stronger condition for the decomposability of connected sets. It is sketched in Fig. 6.

(c4) Let $Z \in \mathfrak{C}$, $Z = A\dot{\cup}B$ a proper bipartition, and A_ι, $B_\jmath$ the connected components of A and B. Then for every $\iota \in I$ there is a $\jmath \in J$ such that $A_\iota \cup B_\jmath \in \mathfrak{C}$.

An immediate consequence of (c4) is

Fact 6.10. *Suppose (c0), (c1), (c2), and (c4) holds. Let $Z \in \mathfrak{C}$ and suppose $Z = A\dot{\cup}B$. Then there is $Z' \subseteq Z$, $Z' \in \mathfrak{C}$ such that both $A' = Z' \cap A \in \mathfrak{C}$ and $B' = Z' \cap B \in \mathfrak{C}$.*

Fact 6.11. *In an integral connectivity space, (c4) implies (c4a).*

Proof. Suppose $Z \in \mathfrak{C}$, $C \subseteq Z$, and W is a connected component of $Z \setminus C$. In the statement of (c4) substitute $Z \setminus C \to A$ and $C \to B$. Then A_ι is W and $B_\jmath$ must be C, hence $C \cup W \in \mathfrak{C}$, i.e., (c4a) holds. □

Lemma 6.12. *Suppose $\mathfrak{S}$ satisfies (S0), (S1), (S2), (S3), and (S4). Then the corresponding integral connecitivity space satisfies (c4).*

Proof. Suppose $Z = A\dot{\cup}B$ and $Z \in \mathfrak{C}$. Since (S0), (S1), (S2) and (S3) are equivalent to (c0), (c1), and (c2), we can partition A and B into their connected components $A = \bigcup_{\iota \in I} A_\iota$ and $B = \bigcup_{\jmath \in I} B_\jmath$. By (SR3) we have $(A_\iota, A \setminus A_\iota) \in \mathfrak{S}$ for all $\iota \in I$.

Now suppose $(A_\iota, B) \in \mathfrak{S}$. Then (S4) implies $(A_\iota, (A \setminus A_\iota) \cup B) = (A_\iota, Z \setminus A_\iota) \in \mathfrak{S}$ contradicting the connectivity of Z. Thus $(A_\iota, B) \notin \mathfrak{S}$.

By Fact 4.8 there is a connected set Z' in $A_\iota \cup B$ intersecting both A_ι and B, i.e., Z' intersects a subset of the connected components of B, say the one indexed by $J' \subseteq J$. By (c1), the union $Z'' = A_\iota \cup B'$, where $B' = \bigcup_{\jmath \in J'} B_\jmath$, is also $\mathcal{C}$-connected. By Fact 2.5, the connected components of B' are exactly the $B_\jmath$, $\jmath \in J'$.

For each connected component $B_\jmath$ of B' we have $(B_\jmath, B' \setminus B_\jmath) \in \mathfrak{S}$. Suppose $(B_\jmath, A_\iota) \in \mathfrak{S}$. Then (S4) implies $(B_\jmath, (B' \setminus B_\jmath) \cup A_\iota) = (B_\jmath, Z'' \setminus B_\jmath) \in \mathfrak{S}$ contradicting the connectedness of Z''. Thus $(A_\iota, B_\jmath) \notin \mathfrak{S}$ and by Fact 4.8 there is a connected set in $A_\iota \cup B_\jmath$ intersecting both A_ι and $B_\jmath$. Since A_ι and $B_\jmath$ are connected, $A_\iota \cup B_\jmath \in \mathcal{C}$. □

Lemma 6.13. *Let $(X, \mathcal{C})$ be a finite integral connectivity space. Then (c4a) and (c4) are equivalent.*

Proof. By Fact 6.11 we only need to prove that (c4a) implies (c4).

Let $Z \in \mathcal{C}$ and $Z = A\dot{\cup}B$. Denote by A_i and B_j the connected components of A and B, resp. We proceed by induction in the number n of connected components of A. For $n = 1$, (c4a) implies $A_1 \cup B_j \in \mathcal{C}$ for all j. The induction hypothesis stipulates that, if A has at most n connected components, then for each component A_i there is a component B_j such that $A_i \cup B_j \in \mathcal{C}$.

Now suppose A has $n + 1$ connected components. Fix a component A_k and consider $Z' = Z \setminus A_k$. Since $B' = Z' \cap B = B$, the connected components of B' are exactly the B_k. The connected components of $A' = Z' \cap A = A \setminus A_k$ are exactly the connected components of A except A_k. Case 1: Z' is connected. Then by induction hypothesis, there is a B_j for each of the n connected components A_i, $i \neq k$, of A' such that $A_i \cup B_j \in \mathcal{C}$. Case 2: Z' is not connected. Then let W be a connected component of Z'. Since each connected component of A' and B' is either contained in W or disjoint from W by Fact 2.5, W is a union of components A_i, $i \neq k$, and B_j. By (c4a) $A_k \cup W \in \mathcal{C}$. W cannot be contained in A' since this would imply $A_k \cup W$ is a connected subset of A, contradicting the fact that A_k is a connected component of A. If $W \subseteq B$, then W coincides with a connected component B_l of B and $A_k \cup B_l \in \mathcal{C}$. In the remaining case W consists of a non-empty subset of the connected components of A_i of A and B_j of B, which are exactly the connected components of $A \cap W$ and $B \cap W$. In particular, $A \cap W$ contains at most n connected components A_i since $A_k \cap W = \emptyset$ by construction. In summary, for all connected components of A with the possible exception of A_k there is a connected component B_j of B so that $A_i \cup B_j \in \mathcal{C}$. Since the choice of A_k was arbitrary, we repeat the argument with a different fixed choice $A_{k'}$ to demonstrate that there is a connected component B_l of B such that $A_k \cup B_l \in \mathcal{C}$. Thus the induction hypothesis holds for $n + 1$ connected components of A, and thus (c4) is satisfied. □

It remains open whether (c4) and (c4a) are equivalent in general. We suspect that this is not the case, although we have not been able to construct a counterexample.

Theorem 6.14. *Let $(X, \mathcal{C})$ be an integral connectivity space and let $\mathfrak{S}_\mathcal{C}$ be the corresponding symmetric separation. Then (c4) is equivalent to (S4).*

Proof. We already know from Lemma 6.12 that (S4) implies (c4). Thus assume that (c4) holds and there are three sets $A, B, C \in X$ such that $(A, B) \in \mathfrak{S}$, $(A, C) \in \mathfrak{S}$ but $(A, B \cup C) \notin \mathfrak{S}$. We aim to show that this is impossible.

From $(A, B \cup C) \notin \mathfrak{S}$ and Fact 4.8 we infer there is $Z \in \mathcal{C}$ such that $Z \subseteq A \cup B \cup C$, $A \cap Z \neq \emptyset$ and $Z \cap (B \cup C) \neq \emptyset$. As a consequence of (c4), furthermore, we can choose Z such that $A' := Z \cap A \in \mathcal{C}$ and $Z \cap (B \cup C) \in \mathcal{C}$. We set $B' = B \cap Z$, and $C' = C \cap Z$. From the assumptions $(A, B) \in \mathfrak{S}$ and $(A, C) \in \mathfrak{S}$ we known that Z cannot be contained in either $A \cup B$ or $A \cup C$. Thus, $B' \neq \emptyset$ and $C' \neq \emptyset$.

From $(A, B) \in \mathfrak{S}$ and (S2) implies $(A', B') \in \mathfrak{S}$. Since $A' \in \mathcal{C}$ by construction, we conclude that A' is a connected component of $(A' \cup B')$. Thus (c4) implies that there is a connected component C_ι of C' such that $A' \cap C_\iota \in \mathcal{C}$. However, $(A, C) \in \mathfrak{S}$ implies $(A', C') \in \mathfrak{S}$ and thus also $(A', C_\iota) \in \mathfrak{S}$, a contradiction. □

We close this section by briefly investigating the impact of additivity on closure spaces.

Theorem 6.15. *If $(X, \mathcal{C})$ is an integral connectivity space satisfying (c4). Then (c3) implies (cX).*

Proof. Let $Z \in \mathcal{C}$ and $A \dot\cup B = Z$. By Fact 6.10 Z contains a connected set Z' such that $A' = Z' \cap A$ and $B' = Z' \cap B$ are $\mathcal{C}$-connected. By (c3) there is a $z \in A' \cup B' = Z' \subseteq Z$ such that $A' \cup \{z\} \in \mathcal{C}$ and $B' \cup \{z\} \in \mathcal{C}$. Either $A' \cup \{z\} \neq A'$ or $B' \cup \{z\} \neq B'$, i.e., the assertion of (cX) holds. □

We close this section by linking (S4) with the corresponding property of the Wallace closure.

Theorem 6.16. [55] *If $\mathfrak{S}$ is a symmetric separation and satisfies (SX), i.e., the corresponding isotone closure function w is pointwise symmetric (R0), then $\mathfrak{S}$ is additive (S4) iff (X, w) is sub-additive (K3).*

Proof. Let $x \in A$ and $B, C \subseteq X$ such that $(\{x\}, B \cup C) \notin \mathfrak{S}$. By (S4) $(\{x\}, B) \notin \mathfrak{S}$ or $(\{x\}, C) \notin \mathfrak{S}$. By definition of w, thus $w(B \cup C) \subseteq w(B) \cup w(C)$, i.e., w satisfies (K3).

Conversely, suppose c is subadditive. Suppose $(A, B) \in \mathfrak{S}$, i.e., $A \cap c(B) = B \cap c(A) = \emptyset$ and $(A, C) \in \mathfrak{S}$, i.e., $A \cap c(C) = C \cap c(A) = \emptyset$. Thus $c(A) \cap (B \cup C) = \emptyset$ and $A \cap (c(B) \cup c(C)) = \emptyset$. Since $c(B \cup C) \subset c(B) \cup c(C)$ by (K3), we also have $A \cap c(B \cap C) = \emptyset$, and thus $(A, B \cup C) \in \mathfrak{S}$. □

Additive integral connectivity spaces that satisfy (c3) are therefore equivalent to the pretopological spaces that satisfy (SR1) and (SR2). It remains open whether the “connectologies” defined by [19], i.e., the integral connectivity spaces satisfying (c3) and (c4a) coincide with these particular pretopological spaces or whether they form a more general class.

7. Some Additional Properties of Interest

7.1. Idempotency

Several axioms are of interest in the context of proximity spaces, see e.g. [53, 63]. These might also be of interest in the more general setting of Wallace separation spaces and (integral) connectivity spaces.

Of particular interest are axioms that render the corresponding Wallace function idempotent:

(C) $(x, A) \in \mathfrak{S}$ implies $(x, w(A)) \in \mathfrak{S}$.
(S5) $(\{x\}, B) \notin \mathfrak{S}$ and $(\{y\}, C) \notin \mathfrak{S}$ for all $y \in B$ implies $(\{x\}, C) \notin \mathfrak{S}$.
(LO) $(A, B) \notin \mathfrak{S}$ and $(\{y\}, C) \notin \mathfrak{S}$ for all $y \in B$ implies $(A, C) \notin \mathfrak{S}$.
(L) $(A, B) \in \mathfrak{S}$ if and only if $(w(A), w(B)) \in \mathfrak{S}$.

Lemma 7.1. [55, Thm.4] *Suppose $\mathfrak{S}$ is a Wallace separation satisfying (SX). Then w is idempotent (K4) if and only if $\mathfrak{S}$ satisfies (S5).*

Axiom (C) was introduced as axiom S.VI. already in [12]. It can replace (S5) as a condition for idempotency of w if $\mathfrak{S}$ is an additive Wallace separation, i.e., if (S0), (S1), (S2), (S3), and (S4) holds [64]. The Lodato axiom (LO) [65] specializes (by replacing A with the singleton set $\{x\}$) to (S5). Since $(\{x\}, C) \notin \mathfrak{S}$ for $x \in A$ implies $(A, C) \notin \mathfrak{S}$, we conclude that (LO) implies (S5). Using equ.(4.1), (S5) can be rewritten as "$B \subseteq w(C)$ implies $w(B) \subseteq w(C)$" and setting $B := w(C)$ yields "$w(w(C)) \subseteq w(C)$" [66, Thm. 1.4]. Thus, a Wallace separation satisfying (S5) has an idempotent Wallace function. (LO) is equivalent to (L) in an additive Wallace separation [66, Thm. 1.5].

7.2. Separation Axioms

Separation axioms can be naturally phrased in terms of $\mathfrak{S}$. Since we pre-suppose symmetry, i.e., (R0), the weakest meaningful separation axioms are

(T1) $(\{x\}, \{y\}) \in \mathfrak{S}$ whenever $x \neq y$. Equivalently. $\{x, y\} \notin \mathfrak{C}$ if $x \neq y$.
(T2) For every two distinct points $x, y \in X$ there are sets $U, V \in 2^X$ such that $U \cup V = X$, $(\{x\}, U) \in \mathfrak{S}$, and $(\{y\}, V) \in \mathfrak{S}$.

Fact 7.2. *Let $(X, \mathfrak{S})$ be a Wallace separation space. Then (T2) implies (T1).*

Proof. Suppose (T2) holds. Then $(\{x\}, U) \in \mathfrak{S}$ and (S3) implies $x \notin U$ and hence $x \in V$. Similarly, $y \in U$. From $(\{y\}, V) \in \mathfrak{S}$ and heredity (S2) we conclude $(\{y\}, \{x\}) \in \mathfrak{S}$. Symmetry (S1) now implies (T1). □

Fact 7.3. *Let $(X, \mathfrak{S})$ be a Wallace separation space satisfying (T1) or (T2), respectively. Then the corresponding closure space (X, w) satisfies the well-known Fréchet (T_1) or Hausdorff (T_2) separation axioms, respectively.*

Proof. Axiom (T1) implies that $y \notin w(\{x\})$ for $y \neq x$, i.e., $w(\{x\}) = \{x\}$ for all $x \in X$. This is one of the many equivalent versions of the Fréchet separation axiom.

Now suppose (T2) holds. Set $N := X \setminus U$ and $M := X \setminus V$. Since $U \cup V = X$ we have $N \cap M = \emptyset$. Furthermore, $(\{x\}, X \setminus M) \in \mathfrak{S}$ is equivalent to $x \notin w(X \setminus M)$, i.e., $x \in X \setminus w(X \setminus M) = i_w(M)$, the interior of M, hence M is a neighborhood of x. One shows analogously that N is a neighborhood of y, i.e., any pair of disjoint points x and y has disjoint neighborhoods in (X, w). This is the usual phrasing for the Hausdorff separation axiom. □

7.3. Haralick's Axiom

In [67] Haralick investigated symmetric separations that satisfy the axiom

(SH) $(\{x\}, \{y\}) \in \mathfrak{S}$ for all $x \in A$ and $y \in B$ implies $(A, B) \in \mathfrak{S}$.

As noted in [26], (SH) implies (SR1) and (SR2) in the disjunctive setting. This remains true in general. Since a separation is by definition hereditary, we have

Fact 7.4. *A symmetric separation satisfies (SH) if and only if it satisfies the following, stronger, version of Haralick's axiom:*

$(\{x\}, \{y\}) \in \mathfrak{S}$ *holds for all* $x \in A$ *and* $y \in B$ *if and only if* $(A, B) \in \mathfrak{S}$.

Lemma 7.5. *Suppose* $\mathfrak{S}$ *is a symmetric separation. Then (SH) implies (SX), (SR0), (SR1), (SR2), and (S4).*

Proof. (SX). Suppose $(\{x\}, B) \in \mathfrak{S}$ for all $x \in A$ and $(A, \{y\}) \in \mathfrak{S}$ for all $y \in B$. By heredity $(\{x\}, \{y\}) \in \mathfrak{S}$ for all $x \in A$ and $y \in B$ and therefore (SH) implies $(A, B) \in \mathfrak{S}$.
(SR0-2). Assuming (SH) we observe that $(A \cup B \cup C, A) \in \mathfrak{S}$ is equivalent to $(A, A) \in \mathfrak{S}$, $(A, B) \in \mathfrak{S}$, and $(A, C) \in \mathfrak{S}$. On the other hand $(A \cup B, A \cup C) \in \mathfrak{S}$ is equivalent to $(A, A) \in \mathfrak{S}$, $(A, B) \in \mathfrak{S}$, $(A, C) \in \mathfrak{S}$, and $(B, C) \in \mathfrak{S}$. Thus $(A \cup B \cup C) \in \mathfrak{S}$ and $(B, C) \in \mathfrak{S}$ holds, as postulated by (SR0-2), if and only if $(A \cup B, A \cup C) \in \mathfrak{S}$.
(SR1). $(A, B) \in \mathfrak{S}$ and $(A \cup B, C) \in \mathfrak{S}$ implies $(\{x\}, \{y\}) \in \mathfrak{S}$ for all $x \in A$ and $y \in B$ and all $x \in A$ and $y \in C$, and therefore also for $x \in A$ and all $y \in B \cup C$. By (SH) this implies $(A, B \cup C) \in \mathfrak{S}$.
(SR2). Consider a family $(A_i, B_i) \in \mathfrak{S}$, $i \in I$. If $x \in \bigcap_{i \in I} A_i$ then $(\{x\}, B_i)$ for all $i \in I$ and thus $(\{x\}, \{y\}) \in \mathfrak{S}$ for all $y \in \bigcup_{i \in I} B_i$. (SH) now implies $(\bigcap_{i \in I} A_i, \bigcup_{i \in I} B_i) \in \mathfrak{S}$.
(SR2) implies (SR0-1), which together with (SR0-2) is equivalent to (SR0) by Lemma 3.17.
(S4). $(A, B) \in \mathfrak{S}$ and $(A, C) \in \mathfrak{S}$ implies $(\{x\}, \{y\}) \in \mathfrak{S}$ for all $x \in A$ and $y \in B \cup C$. Now (SH) implies $(A, B \cup C) \in \mathfrak{S}$. □

As (SH) implies (SX), we can equivalently express (SH) and (S2) in terms of the Wallace function. Using equ.(4.1) we can recast Fact 7.4 in the form "$y \notin w(A)$ if and only if $y \notin w(\{x\})$ for all $x \in A$". This is in turn equivalent to "$y \in w(A)$ iff $y \in w(\{x\})$ for all $x \in A$", and hence $w(A) = \bigcup_{x \in A} w(\{x\})$. The axiom

(K5) $$c(A) = \bigcup_{x \in A} c(\{x\})$$

is known as *total additivity* in the context of closure spaces. In the topological literature such spaces are often called *Alexandroff spaces*. Obviously (K5) implies (K0), (K1), and (K3). It is shown e.g. in [68, 69] that Alexandroff closure spaces are equivalent to binary relations. A related result is Prop. 4.2 of [26], which characterizes Haralick separations as exactly those that are generated from a set $\mathcal{Q}$ of pairs of points that are considered as connected. This class, in particular, includes graphs.

Alexandroff topologies, i.e., closure spaces that satisfy (K0) through (K5), are equivalent to binary dominance (transitive and reflexive) relations [70].

The Haralick axiom (SH) suggests the consider also a stronger condition for the self-separated points:

(SH0) $(\{x\},\{x\}) \in \mathfrak{S}$ implies $(\{x\},\{y\}) \in \mathfrak{S}$ for all $y \in X$.

(SH0) obviously implies (SR0). The following example shows that (SH) and (SR0) are not sufficient to imply (SH0):

Example. Let $X = \{a, b, c\}$ and $\mathfrak{S}$ consist of $(\{a\},\{a\})$, $(\{b\},\{c\})$ and $(\{c\},\{b\})$ as well $(A, \emptyset)$ and $(\emptyset, A)$ for all $A \subseteq X$. (SH) and (S2) are clearly satisfied. We have $\{a, b, c\}^\circ = \emptyset$, $\{a, b\}^\circ = \emptyset$, $\{a, c\}^\circ = \emptyset$, $\{b, c\}^\circ = \emptyset$, hence (SR0) is satisfied trivally. On the other hand, we have $\mathring{X} = \{a\}$. Hence (SH0) implies that we should also have e.g. $(\{a, b\},\{a, c\}) \in \mathfrak{S}$.

If (SH) and (SH0) holds we obtain a particularly simple representation for

$$\mathring{Y} = \{y \in Y | (\{y\},\{y\}) \in \mathfrak{S}\} = Y \cap \mathring{X} \,. \tag{7.1}$$

Hhere $\mathring{X}$ is the set of points that do not belong to a connected component of the entire space. Equivalently, in such a space $\mathring{X}$ can be seen as the set of points not connected to themselves.

7.4. Efremovič's Axiom

Proximity is the complement of separation, i.e., two sets $A, B \in 2^X$ are interpreted as "near" each other iff $(A, B) \notin \mathfrak{S}$. A proximity space in the sense of Efremovič [71] is equivalent to an additive Wallace separation that in addition satisfies the separation property (T1) and the axiom

(S6) $(A, B) \in \mathfrak{S}$ implies that there is $U \subseteq X$ such that $(A, U) \in \mathfrak{S}$ and $(X \setminus U, B) \in \mathfrak{S}$.

Instead of Efremovič's axiom (S6) the following *condition of normality* is often used in the literature:

(S6') $(A, B) \in \mathfrak{S}$ implies that there are sets $U, V \in 2^X$ such that $U \cup V = X$, $(A, U) \in \mathfrak{S}$ and $(B, V) \in \mathfrak{S}$.

Fact 7.6. *If $(X, \mathfrak{S})$ is a Wallace separation space then (S6) and (S6') are equivalent.*

Proof. (S6') is obtained from (S6) by setting $V := X \setminus U$, i.e., (S6) implies (S6'). Now suppose (S6') and pick $U' \subseteq U$ and $V' \subseteq V$ with $U' \cap V' = \emptyset$ and $U' \cup V' = X$, i.e., $V' = X \setminus U'$. By heredity we have $(A, U') \in \mathfrak{S}$ and $(B, V') \in \mathfrak{S}$ as desired. □

In an Efremovič proximity space, the Wallace function is idempotent (and hence defines a topology). Furthermore, it is well known that in a topological space, i.e., when $\mathfrak{S}$ satisfies (S0) through (S5) as well as (SX), then (S6) is equivalent to complete regularity [53]. In general, the Wallace function of a proximity space, which satisfies (S0) through (S4) and (S6) is completely regular. The implication (S6) $\Rightarrow$ (LO) in additive Wallace separations is proved e.g. in [72, Thm. 2.7].

8. Catenous Functions

Let $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a function between two connectivity spaces. Following [19] we say that f is *catenous* if $A \in \mathcal{C}_X$ implies $f(A) \in \mathcal{C}_Y$ for all $A \in 2^X$. It follows directly from the definition that the concatenation of catenous functions is again catenous.

Denote by $\not\models$ the totally disconnected space on two points, i.e., the space $\{0, 1\}$ in which only $\emptyset$, $\{0\}$, and $\{1\}$ are connected. Note that we may regard $\not\models$ also as neighborhood space since $w(\emptyset) = \emptyset$, $w(\{0\}) = \{0\}$, $w(\{1\}) = \{1\}$, and $w(\{1, 2\}) = \{1, 2\}$.

A classical theorem in point set topology asserts that X is connected if and only if every continuous function $X \to \not\models$ is constant. In the realm of connectivity space we have the obvious analog:

Fact 8.1. *An integral connectivity space* $(X, \mathcal{C})$ *is connected if and only if every catenous function* $f : (X, \mathcal{C}) \to \not\models$ *is constant.*

If $A \subseteq X$ is a connected set and f is not constant on A, then $f(A) = \{0, 1\}$, i.e., not connected, contradicting that f is catenous. Thus f must be constant on every connected set. Any function $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ that is constant on connected components of X is obviously catenous as long as $(Y, \mathcal{C}_Y)$ is an integral connectivity space.

The restriction of a catenous function to a subspace remains catenous. Interestingly, continous functions between topological spaces are catenous, but the converse is not true in general [19].

9. Summary and Concluding Remarks

In this contribution we have summarized a variety of independent approaches to axiomatizing connectedness in point set topology. We have focussed in particular on the close relationship between separation spaces *sensu* Wallace and direct axiom systems for connected sets. Extending prior work by Christian Ronse we characterize the subclass of separation spaces that are equivalent to connectivity spaces. These separations satisfy quite restrictive conditions that encapsulate that connected components are properly separated from each other. Generalized closure spaces form an important special subclass, characterized by a single axiom (SX).

Table 1. Summary of axiom systems. Properties of $(X, \mathfrak{S})$ and $(X, \mathcal{C})$ are equivalent if $(X, \mathfrak{S})$ satisfies in addition (SR0), (SR1), and (SR2). Likewise, properties of $(X, \mathfrak{S})$ and (X, c) are equivalent, if $(X, \mathfrak{S})$ satisfies (SX). Equivalence of (S4) and (c4) in addition requires (c0) to (c2) or (S0) to (S3).

$(X, \mathcal{C})$ (SR0), (SR1), (SR2)	$(X, \mathfrak{S})$	(X, c) (SX)
	(S0), (S2)	(K0), (K1)
	(S1)	(R0)
(c0), (c1)	(S0), (S1), (S2)	(K0), (K1), (R0)
(c2)	(S3)	(K2)
(c4)	(S4)	(K3)
(c2), (c3), (c4)	(S3), (S4), (SX)	pretopology
–	(S5)	(K4)
graph-like	(SH)	(K5)
(T1)	(T1)	Fréchet
–	(T2)	Hausdorff
–	(S6)	completely regular

Separations (and the the equivalent proximities) are thus strictly more expressive than either closure spaces or connectivities.

Key properties of the closure (Wallace function) track the most important characteristics of separation functions, see Tab. 1. Although the axiom systems for separations and closures have been devised in often quite different contexts, they closely match and become equivalent in suitably restricted settings. In particular, Wallace separations and integral connectivity spaces correspond to neighborhood spaces, while additive Wallace separations and additive connectivity spaces generalize pretopologies.

The present contribution primarily aims at collecting and integrating the available basic results on generalized connectivity structures. Along the way, interesting research questions have appeared. For instance, it seems worthwhile to investigate generalizations of axioms of separations and regularity to connectivity spaces and their separation relations in such a way that the Wallace function w has prescribed separation or regularity properties [73, 74]. Ronse's axioms (SR1) and (SR2) do not seem to have been studied in any depth in the context of closure spaces. It is likely that more elegant characterizations of those neighborhood spaces, pretopologies, and topologies can be found, whose closure functions are completely determined by the connected sets. Furthermore, we have not discussed important constructions such as product spaces. In many details and combinations of axioms, finally, relationships between connectivity spaces, separation relations, and Wallace functions remain to be elucidated.

Originally, this work was motivated by the realization that connectedness – rather than other topological constructions such as closure or separation – is the key ingredient for understanding fitness landscapes. In this context, as in the case of image processing, the special case of a finite set X is of most direct interest. We therefore close this contribution with a brief discussion of finite integral connectivity spaces: For finite X, (c4a) and (c4") are obviously equivalent, and hence every connected set Z is a union of finitely many disjoint singletons, whence we can break it down into connected pairs. Note also that (S4) and (SH) become equivalent as an immediate consequence of finiteness. A finite additive connectivity space therefore is a finite graph Γ with vertex set X and edges defined as the two-element subsets of $\mathcal{C}$. Property (cX) is also satisfied automatically. The Wallace function of a singleton $\{x\}$ is thus simply its graph theoretic neighborhood, $w(\{x\}) = \{y \in X|\{x, y\} \in \mathcal{C}\}$. Additivity thus implies that $w(A) = \bigcup_{x\in A} w(\{x\})$. Since additive connectivity spaces are simply the finite undirected graphs, there is little to be gained by starting with abstract connectedness.

As the example of recombination-based search spaces in [36] shows, however, much less obvious structures arise when additivity cannot be assumed, see also [44]. Such non-additive concepts of connectedness arise in particular in the context of combinatorial optimization when population-based heuristics implicitly define the structure of the search space, but also in the context of convexities, which in general also lack additivity.

The current setting of connective spaces still feels too restricted at times and not all concepts of "connectivity" that appear in the graph theory literature can be accomodated in the present setting. Higher-order edge and vertex connectivity, for instance, requires modification of axiom (c1). Instead of the intersection of two sets, more sophisticated "overlap criteria" need to be defined. Explorations in this direction can be found e.g. in work of Serra [75], Braga-Neto [33], and Wilkinson [76]. The resulting structures generalize the connectivity openings of Section 2.3 and for instance define "hyperconnected components" that are non-overlapping but not necessarily disjoint. Directed versions of reachability or accessibility also play an important role in theoretical biology e.g., in models of phenotypic evolution [77]. Most mathematical work in this direction is based on generalized closure functions without assuming (R0). This does not give rise to a non-symmetric notion of connectedness, however. An axiomatic approach towards directed connectivity in general, with a focus on possible application in image analysis is given in [78]. It will be interesting to see if and how such generalized structures interrelate with classical concepts of point set topology.

Acknowledgements

We thank an anonymous referee for extensive feedback on early versions of this manuscript. This work was supported in part by the *Deutsche Forschungsgemeinschaft* under the auspices of SPP-1590.

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Bärbel M. R. Stadler
Max Planck Institute for Mathematics in the Sciences
Inselstraße 22
D-04103 Leipzig,
Germany

e-mail: `baer@bioinf.uni-leipzig.de`

Peter F. Stadler
Bioinformatics Group,
Department of Computer Science; and Interdisciplinary Center for Bioinformatics,
University of Leipzig,
Härtelstrasse 16-18, D-04107 Leipzig,Germany

Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103 Leipzig, Germany

RNomics Group, Fraunhofer Institut für Zelltherapie und Immunologie, Deutscher Platz 5e, D-04103 Leipzig, Germany

Department of Theoretical Chemistry, University of Vienna, Währingerstraße 17, A-1090 Wien, Austria

Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe, NM87501, USA
e-mail: `studla@bioinf.uni-leipzig.de`