# Diagonalized Cartesian Products of S-prime graphs are S-prime


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**Abstract**

A graph is said to be S-prime if, whenever it is a subgraph of a nontrivial Cartesian product graph, it is a subgraph of one of the factors. A diagonalized Cartesian product is obtained from a Cartesian product graph by connecting two vertices of maximal distance by an additional edge. We show there that a diagonalized product of S-prime graphs is again S-prime. Klavžar *et al.* [*Discr. Math.* **244**: 223-230 (2002)] proved that a graph is S-prime if and only if it admits a nontrivial path-$k$-coloring. We derive here a characterization of all path-$k$-colorings of Cartesian products of S-prime graphs.


## 1 Introduction and Preliminaries

A graph $S$ is said to be *S-prime* ($S$ stands for "subgraph") w.r.t. to an arbitrary graph-product $\star$ if for all graphs $G$ and $H$ with $S \subseteq G \star H$ holds: $S \subseteq H$ or $S \subseteq G$. A graph is *S-composite* if it is not S-prime. The class of S-prime graphs was introduced and characterized for the direct product by Gert Sabidussi in 1975 [10]. He showed that the only S-prime graphs with respect to the direct product are complete graphs or complete graphs minus an edge. Analogous notions of S-prime graphs with respect to other products are due to Lamprey and Barnes [8, 9]. They showed that the only S-prime graphs w.r.t. the strong product and the lexicographic product are the single vertex graph $K_1$, the disjoint union $K_1 \cup K_1$ and the complete graph on two vertices $K_2$. Moreover, they characterized S-prime graphs w.r.t. the Cartesian product.

We consider finite, simple, connected and undirected graphs $G = (V,E)$. A graph $H$ is a subgraph of a graph $G$, in symbols $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We will be concerned here with the Cartesian product $G \Box H$. It has vertex set $V(G \Box H) = V(G) \times V(H)$; two vertices $(g_1,h_1)$, $(g_2,h_2)$ are adjacent in $G \Box H$ if $(g_1,g_2) \in E(G)$ and $h_1 = h_2$, or $(h_1,h_2) \in E(G_2)$ and $g_1 = g_2$. A Cartesian product $G \Box H$ is called *trivial* if $G \simeq K_1$ or $H \simeq K_1$. A graph $G$ is *prime* with respect to the Cartesian product if it has only a trivial Cartesian product representation. For detailed information about product graphs we refer the interested reader to [4] and [5].

In the following we will consider the Cartesian product only. Therefore, the terms S-prime and S-composite refer to this product from here on.

S-prime graphs can be characterized in terms of *basic S-prime graphs* [8, 9]. Following [8, 9], we define basic S-prime graphs recursively: An S-prime graph is basic if it has at least three vertices and contains no proper basic S-prime subgraphs. Moreover, these same authors showed that every S-prime graph is either a basic S-prime graph or can be obtained from basic S-prime graphs using two special operations. The only basic S-prime graphs with less than 7 vertices are $K_3$ and $K_{2,3}$. Other examples of S-prime graphs include the complete graphs $K_n$ with $n \geq 1$ vertices and the complete bipartite graphs $K_{m,n}$ with $m \geq 2, n \geq 3$. Not much is known, however, about the structure of (basic) S-prime graphs, although Klavžar *et al.* [6, 7] and Brešar [1] proved several characterizations of (basic) S-prime graphs. For our purposes, the characterization of S-composite graphs in terms of particular colorings [6] is of most direct interest.

Before we proceed, we introduce some notation. Given $G = (V,E)$, we will write $G \ddagger (uv)$ for the graph with vertex set $V$ and $E \ddagger (u,v)$ for each of the set operations $\ddagger \in \{\setminus, \cup, \cap\}$.

The Cartesian product is associative. Therefore, a vertex $x$ of a Cartesian product $\Box_{i=1}^n G_i$ is properly "coordinatized" by the vector $c(x) := (c_1(x), \ldots, c_n(x))$ whose entries are the vertices $c_i(x)$ of its factor graphs $G_i$. Two adjacent vertices in a Cartesian product graph therefore differ in exactly one coordinate. In a Cartesian product $\Box_{i=1}^n G_i$, the induced subgraph $G_j^x$ with vertex set $\{(c_1(x), \ldots c_{j-1}(x), v, c_{j+1}(x), \ldots, c_n(x)) \in V(G) \mid v \in V(G_j)\}$ is isomorphic to the factor $G_j$ for every $x \in V(G)$. We call this subgraph a $G_j$-layer. Throughout this contribution we will use $I_n = \{1, \ldots, n\}$ to index the factors.

A *k-coloring* of $G$ is a surjective mapping $F : V(G) \to \{1, \ldots, k\}$. This coloring need not be proper, i.e., adjacent vertices may obtain the same color. A path $P$ in $G$ is *well-colored* by $F$ if for any two consecutive vertices $u$ and $v$ of $P$ we have $F(u) \neq F(v)$. Following [6], we say that $F$ is a *path-k-coloring* of $G$ if $F(u) \neq F(v)$ holds for the endpoints of every well-colored $u,v$-path $P$ in $G$. For $k = 1$ and $k = |V|$ there are trivial path-$k$-colorings: For $k = 1$ the coloring is constant and hence there are no well-colored paths. On the other hand if a different color is used for every vertex, then every path, of course, has distinctly colored endpoints. A path-$k$-coloring is nontrivial if $2 \leq k \leq |V(G)| - 1$.

**Theorem 1** ([6])**.** *A connected graph $G$ is S-composite if and only if there exists a nontrivial path-$k$-coloring.*

The next corollary, which follows directly from Theorem 1, will be useful in the subsequent discussion.

**Corollary 1.** *Consider an S-prime graph $S$ and let $F$ be a path-$k$-coloring of $S$. If there are two distinct vertices $u,v \in V(S)$ with $F(u) = F(v)$ then $F$ is constant, i.e., $k = 1$.*

Now consider a product graph $\Box_i G_i$. We say that all vertices *within* the $G_j$-layer $G_j^x$ have the same color if $F(a) = F(b)$ holds for all vertices $a,b \in V(G_j^x)$. Note that this does not imply that vertices of different $G_j$-layer obtain the same color.

The main topic of this contribution are *diagonalized* Cartesian product graphs.

**Definition 1.** *A graph $G$ is called a* diagonalized *Cartesian product, if there is an edge $(u,v) \in E(G)$ such that $H = G \setminus (uv)$ is a nontrivial Cartesian product and $u$ and $v$ have maximal distance in $H$.*

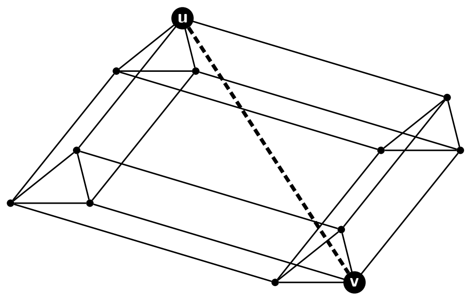


Figure 1: A diagonalized Cartesian Product of the graph $K_2 \Box K_2 \Box K_3$.

For an example of a diagonalized Cartesian product see Figure 1. Clearly, diagonalized Cartesian products need not be basic S-prime graphs because Cartesian products of basic S-prime graphs contain basic S-prime graphs as their layers. Likewise, diagonalized Cartesian products of $K_2$'s, i.e., diagonalized hypercubes, are not basic S-prime graphs in general, even though the graph $K_2$ is not itself a basic S-prime graph. As an example consider the diagonalized hypercube $Q_2$ and $Q_3$ that contain $K_3$ and $K_{2,3}$ as subgraphs, respectively, see Figure 2. Furthermore, there are families of (basic) S-prime graphs that are not diagonalized Cartesian products, e.g., $K_3$, $K_{2,3}$ and the construction of the graph $A_n$ in [6].

Nevertheless, in this contribution we will show that diagonalized Cartesian products of S-prime graphs are S-prime. Diagonalized Cartesian products of S-prime graphs play a crucial role in the local prime factor decomposition algorithm for strong product graphs, see [3, 2]. Furthermore, we will give a necessary and sufficient condition for $k$-colorings of S-prime graphs to be path-$k$-colorings of Cartesian products of S-prime graphs.

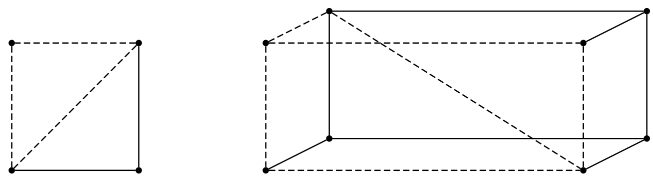

Figure 2: Diagonalized Hypercubes $Q_2$ and $Q_3$ are not basic S-prime, since they contain basic S-prime graphs $K_3$, resp. $K_{2,3}$, highlighted by dashed edges.

## 2 Path-$k$-colorings of Cartesian Products of S-prime graphs

Let us start with a brief preview of this section. We first establish that every nontrivial Cartesian product $G_1 \Box G_2$ has a nontrivial path-$k$-coloring. For instance, choose $k = |V(G_1)|$ and assign to every vertex $x$ with coordinates $(x_1, x_2)$ the color $x_1$.

Given a Cartesian product $G = \Box_{i=1}^{n} S_i$ of S-prime graphs with a nontrivial path-$k$-coloring $F$, first we will show that there is an $S_i$-layer on which $F$ is constant. Next, we prove that is true for all $S_i$-layers. We then proceed to show that $F$ is constant even on any $H$-layer with $H = \Box_{j \in J} S_j$, provided that certain conditions are satisfied. This eventually leads us to necessary and sufficient conditions for path-$k$-colorings. This result, in turn, will be demonstrated to imply that diagonalized Cartesian products of S-prime graphs are S-prime.

We start our exposition with a simple necessary condition:

**Lemma 1.** *Let $H \subseteq G$ and suppose $F$ is a path-k-coloring of $G$. Then the restriction $F_{|V(H)}$ of $F$ on $V(H)$ is a path-k-coloring of $H$. Moreover, if $V(H) = V(G)$ and $F$ is a nontrivial path-k-coloring of $G$, then it is also a nontrivial path-k-coloring of $H$.*

*Proof.* Suppose $H$ is not path-$k$-colored. Then there is a $u,v$-path $P_{u,v}$ in $H$ that is well-colored, but $u$ and $v$ have the same color. This path $P_{u,v}$ is also contained in $G$, contradicting the assumption that $F$ is a path-$k$-coloring of $G$. The second statement now follows directly from $|V(G)| = |V(H)|$. □

**Lemma 2.** *Let $F$ be a nontrivial path-k-coloring of $G$. Then there are adjacent vertices $u, v \in V(G)$ with $F(u) = F(v)$.*

*Proof.* Since $k \leq |V(G)| - 1$ it follows that there are at least two vertices of the same color, say $x$ and $y$. Assume now there is a path $P_{x,y}$ from $x$ to $y$, such that all consecutive vertices have different colors. Then $P_{x,y}$ would be well-colored. But the endpoints of $P_{xy}$ satisfy $F(x) = F(y)$ so that $F$ cannot be a path-$k$-coloring, a contradiction. Thus there are consecutive, and hence adjacent, vertices with the same color. □

For later reference, we state the following observation that can be verified by explicitly enumerating all colorings, see Figure 3 for a subset of cases.

**Lemma 3.** *The hypercube $Q_2 = K_2 \Box K_2$ has no path-3-coloring. In particular, every path-2-coloring of $Q_2$ has adjacent vertices with the same color.*

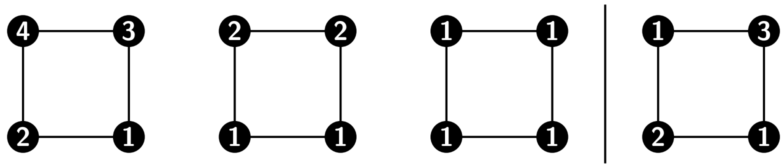


Figure 3: Possible path-$k$-coloring of a square $Q_2$ for $k = 1,2,4$. A possible well coloring that is not a path 3-coloring is shown on the right-hand side graph

We next show that $F$ is constant on each $S_j$-layer whenever there is one $S_j$-layer that contains two distinct vertices with the same color. More precisely:

**Lemma 4.** *Let $G = \Box_{i=1}^{n} S_i$ be a given Cartesian product of S-prime graphs and let $F$ be a nontrivial path-k-coloring of $G$. Furthermore let $u,w \in V(S_j^u)$ be two distinct vertices satisfying $F(u) = F(w)$. Then $F(x) = F(y)$ holds for all vertices $x,y \in V(S_j^b)$ in each $S_j$-layer $S_j^b$.*

*Proof.* Corollary 1 and Lemma 1 imply that all vertices of the layer $S_j^u$ have the same color. For $b \in V(S_j^u)$ there is nothing to show. Thus, assume $b \notin V(S_j^u)$, i.e., $S_j^u \neq S_j^b$, and an arbitrary edge $e = (u,v) \in E(S_j^u)$. Let $\tilde{u} \in V(S_j^b)$ be the vertex with coordinates $c_j(\tilde{u}) = c_j(u)$. Moreover, let $P_{u,\tilde{u}} := (u = u_1, u_2, .., u_l = \tilde{u})$ be a path from $u$ to $\tilde{u}$ such that $c_j(u_k) = c_j(u)$ for all $k = 1,\dots,l$. None of the edges $(u_k, u_{k+1})$ is contained in an $S_j$-layer. By definition of the Cartesian product there is a unique square $(u, u_2, v_2, v)$ where $v_2$ has coordinates $c_i(v_2) = c_i(u_2)$ for $i \neq j$ and $c_j(v_2) = c_j(v)$. Lemma 3 now implies that the only $F$ on the square is either constant or a path-2-coloring, i.e., the assumption $F(u) = F(v)$ implies $F(u_2) = F(v_2)$.

By induction on the length of the path $P_{u,\tilde{u}}$ we see that $F(u_k) = F(v_k)$, whenever $c_i(v_k) = c_i(u_k)$ for all $i \neq j$ and $c_j(v_k) = c_j(v)$. The assumption $\tilde{u} \in V(S_j^b)$ and our choice of the coordinates implies $(u_l, v_l) = (\tilde{u}, v_l) \in E(S_j^b)$. We apply Lemma 3 to the square $(u_{l-1}, \tilde{u}, v_l, v_{l-1})$ with $F(u_{l-1}) = F(v_{l-1})$ to infer $F(\tilde{u}) = F(v_l)$. Corollary 1 and Lemma 1 imply that for all vertices $x,y \in V(S_j^b)$ holds $F(x) = F(y)$. □

It is important to notice that Lemma 4 only implies that $F$ is constant on $S_j$-layers, but it does not imply that all $S_j$-layers obtain the same color.

**Corollary 2.** *Let $G = \Box_{i=1}^{n} S_i$ be a given product of S-prime graphs and let $F$ be a nontrivial path-k-coloring of $G$. Then there is a $j \in I_n$ such that, for every $v \in V(G)$, $F$ is constant on $S_j^v$.*

*Proof.* The assertion follows directly from Lemma 2, Lemma 4, and the definition of the Cartesian product. □

**Lemma 5.** *Let $F$ be a nontrivial path-k-coloring of the Cartesian product $G = \Box_{i=1}^{n} S_i$ of S-prime graphs $S_i$. Let $H = \Box_{j \in J} S_j$ be the product of a subset of factors of $G$, where $J \subseteq I_n$ denotes an arbitrary subset of indices. Moreover, let $H^a$ be an H-layer such that $F$ is constant on $V(H^a)$. Then $F$ is constant within each H-layer.*

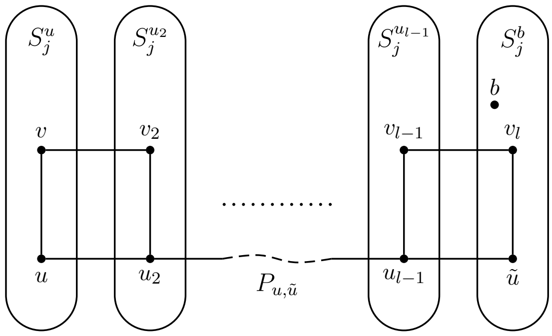


Figure 4: Idea of the proof of Lemma 4. The path $P_{u,\tilde{u}}$ connects vertices $u$ and $u_k$ ($k = 2,\ldots,l$) of distinct $S_j$-layers. If $F(u_{k-1}) = F(v_{k-1})$ then the squares $(u_{k-1},u_k,v_k,v_{k-1})$ located in adjacent $S_j$-layers must admit a path-1-coloring or a path-2-coloring, enforcing that $u_k$ and $v_k$ must have the same color. This, in turn, is used to show that $F$ is constant on the entire layer $S_j^{u_k}$.

*Proof.* Let $H^a$ be an $H$-layer defined as above and assume $H^a \neq H^b$. By assumption, $F$ is constant on $V(H^a)$. Thus $F$ is also constant on each $S_j$-layer $S_j \subseteq H^a$, $j \in J$, and Lemma 4 then implies that $F$ is also constant within every $S_j$-layer with $j \in J$. Now choose two arbitrary vertices $x,y \in V(H^b)$. By connectedness of $H^b$ there is a path $P_{x,y}$ from $x$ to $y$ consisting only of vertices of this $H$-layer $H^b$. Notice that any two consecutive vertices $x_k, x_{k+1} \in P_{x,y}$ are contained in some $S_j$-layer such that $j \in J$ and therefore $F(x_k) = F(x_{k+1})$. Therefore, the coloring $F$ must be constant along $P$, hence $F(x) = F(y)$. Thus $F$ is constant on $V(H^b)$. □

Next we consider two (not necessarily prime) factors $H_1, H_2$ of a Cartesian product of S-prime graphs and ask under which conditions a path-$k$-coloring on $(H_1 \Box H_2)$-layers must be constant.

**Lemma 6.** *Let $F$ be a nontrivial path-$k$-coloring on the Cartesian product $G = \Box_{i=1}^{n} S_i$ of S-prime graphs $S_i$. Let $H_1 = \Box_{j\in J} S_j$ and $H_2 = \Box_{k\in K} S_k$ be two distinct Cartesian products of factors $S_i$ of $G$, where $J,K \subseteq I_n$ and $J \cap K = \emptyset$. Then $F$ is constant on each $(H_1 \Box H_2)$-layer whenever $F$ is constant on some $H_1$-layer $H_1^a$ and on some $H_2$-layer $H_2^b$.*

*Proof.* Let $H_1^a$ and $H_2^b$ as constructed above. Lemma 5 implies that all vertices within each $H_1$ layer and within each $H_2$-layer, resp., have the same color. For all vertices $z \in V(H_1^a)$ there is an $H_2$-layer $H_2^z$, Thus for all vertices $x,y \in V(H_2^z)$ holds $F(x) = F(y) = F(z) = F(a)$. By definition of the Cartesian product, this implies in particular that all vertices within the layer $(H_1 \Box H_2)^a$ have the same color $F(a)$. Hence we can apply Lemma 5 and conclude that all vertices within each $(H_1 \Box H_2)$-layer have the same color. □

Now we are in the position to characterize nontrivial path-$k$-colorings.

**Lemma 7.** *Let $F$ be a nontrivial path-k-coloring of the Cartesian product $G = \Box_{i=1}^{n} S_i$ of S-prime graphs $S_i$, and consider two distinct vertices $u, v \in V(G)$ satisfying $F(u) = F(v)$. Let $J = \{j \mid c_j(u) \neq c_j(v)\} \subseteq I_n$ denote the index set of the coordinates in which $u$ and $v$ differ, and let $H = \Box_{j \in J} S_j$ be the Cartesian product of the corresponding factors $S_j$ of $G$. Then $F$ is constant within each $H$-layer $H^b$.*

*Proof.* First assume that $v \in V(S_l^u)$ for some $l$, which implies that $J = \{l\}$ by definition of the Cartesian product. In this case, the statement follows directly from Lemma 4.
Now assume that there is no $l$ such that $v \in V(S_l^u)$. Lemma 4 and Corollary 2 together imply that there is an index $i$ such that all vertices within each $S_i$-layer have the same color. In particular, this is true for $S_i^u$ and $S_i^v$. Together with Lemma 4, this observation implies that, since $F(u) = F(v)$, $F$ is constant on $V(S_i^u) \cup V(S_i^v)$. Now let $\tilde{u} \in V(S_i^v)$ be the vertex with coordinates $c_i(u) = c_i(\tilde{u})$ and denote by $J_1 = \{j \mid c_j(u) \neq c_j(\tilde{u})\} = J \setminus \{i\}$ the set of indices in which the coordinates of $u$ and $\tilde{u}$ differ. Notice that $J \setminus \{i\} = J$, if $v = \tilde{u}$.
Let $P_{u,\tilde{u}} := (u = u_1, u_2, .., u_k = \tilde{u})$ be a path from $u$ to $\tilde{u}$ such that for all vertices $x \in P_{u,\tilde{u}}$ holds $c_r(x) = c_r(u)$ for all $r \in I_n \setminus J_1$. In other words, no edge of an $S_r$-layer, $r \notin J_1$ is contained in the path $P_{u,\tilde{u}}$, and hence in particular no edge of an $S_i$-layer. From $F(u) = F(\tilde{u})$ and the fact that $G$ is path-$k$-colored, we can conclude that there is an edge $(u_l, u_{l+1}) \in P_{u,\tilde{u}}$ of some layer different from $S_i$ such that $F(u_l) = F(u_{l+1})$.

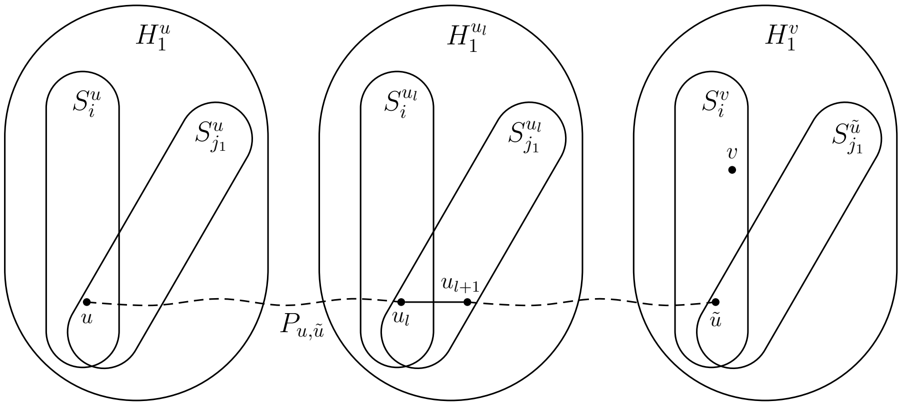


Figure 5: Idea of the proof of Lemma 7. The path $P_{u,\tilde{u}}$ connects a pair of vertices with the same color in $S_i^u$ to $S_i^v$. It therefore must contain two consecutive vertices $u_l$ and $u_{l+1}$ with the same color. It follows that all vertices within the layer $S_i^{u_l}$ and $S_{j_1}^{u_l}$ have the same color $F(u_l)$ and finally one shows that all vertices within each $H_1$-layer with $H_1 = S_i \Box S_{j_1}$ have the same color.

These consecutive vertices $u_l$ and $u_{l+1}$ differ in exactly one coordinate $c_{j_1}$ for some $j_1 \in J_1$, hence $u_l$ and $u_{l+1}$ are contained in some $S_{j_1}$-layer. Lemma 4 implies that all vertices of this layer $S_{j_1}^{u_l}$ and therefore all vertices within each $S_{j_1}$-layer have the same color. Lemma 6 now implies that $F$ is constant on each $H_1$-layer with $H_1 = S_i \Box S_{j_1}$, and in particular, all vertices $x, y \in V(H_1^u) \cup V(H_1^v)$ have the same color, we have again two different layers that have the same color. Just as before we will construct a path between these layers, which implies that the endpoints of this path have the same color. Since $G$ is path-$k$-colored, this path must contain an edge $(u_t, u_{t+1})$ with $F(u_t) = F(u_{t+1})$.

More precisely, let $\tilde{u}$ be a vertex of this new $H_1$-layer $H_1^v$ such that $c_i(\tilde{u}) = c_i(u)$ and $c_{j_1}(\tilde{u}) = c_{j_1}(u)$. Again we choose a Path $P_{u,\tilde{u}}$ constructed as above, where $J_1$ is replaced by $J_2 = J_1 \setminus \{j_1\}$. In other words for all vertices $x \in P_{u,\tilde{u}}$ holds $c_r(x) = c_r(u)$ for all $r \in I_n \setminus J_2$, i.e. in particular no edge of $P_{u,\tilde{u}}$ is contained in any $H_1$-layer. Notice that $|J_2| = |J_1| - 1$. Again we can conclude that there are consecutive vertices $u_t, u_{t+1} \in P_{u,\tilde{u}}$ such that $F(u_t) = F(u_{t+1})$, since $F(\tilde{u}) = F(u)$ and $G$ is path-$k$-colored. Let these consecutive vertices $u_t$ and $u_{t+1}$ differ in coordinate $c_{j_2}$ for some $j_2 \in J_2$. Using the same arguments as before we can infer that all vertices in between each $H_2 = (S_i \Box S_{j_1} \Box S_{j_2})$-layer must have the same color.

Repeating this procedure generates, in each step, a new index set $J_s$ with $|J_s| = |J_{s-1}| - 1$ for $s = 2, \ldots, |J_1|$, and all vertices within each $H_s$-layer with $H_s = S_i \Box \left(\Box_{j \in J_1 \setminus J_s} S_j\right) \Box S_{j_s}$ for some $j_s \in J_s$ are shown to have the same color. For $s^* = |J_1|$ we have $|J_{s^*}| = 1$. Moreover the path $P_{u,\tilde{u}}$ with $c_r(\tilde{u}) = c_r(u)$ for all $r \in I_n \setminus \{j^*\}$ with $j^* \in J_{s^*}$ consists only of vertices that are included in this $S_{j^*}$-layer $S_{j^*}^u$. Since $F(u) = F(\tilde{u})$ and $u, \tilde{u} \in S_{j^*}^u$ we can conclude that all vertices $x \in S_{j^*}^u$ have the same color $F(u)$. From Lemma 5 and Lemma 6 it follows that $F$ is constant on each $H_{s^*}$-layer, where $H_{s^*} = \left(S_i \Box (\Box_{j \in J_1 \setminus J_{s^*}} S_j) \Box S_{j^*}\right)$. Since $\{i\} \cup (J_1 \setminus J_{s^*}) \cup \{j^*\} = \{i\} \cup ((J \setminus \{i\}) \setminus \{j^*\}) \cup \{j^*\} = J$, we conclude that all vertices within each $(\Box_{j \in J} S_j)$-layer have the same color, completing the proof of the lemma. $\Box$

Since two vertices with maximal distance contained in a Cartesian product of nontrivial factors differ in all coordinates we can conclude the following corollary.

**Corollary 3.** *Let $F$ be a path-$k$-coloring of the Cartesian product $G = \Box_{i=1}^n S_i$ of S-prime graphs $S_i$ and suppose $u, v \in V(G)$ are two vertices with maximal $G$-distance that have the same color. Then $F$ is constant on $G$, i.e., $k = 1$.*

# 3 Main Results

We are now in the position to give a complete characterization of path-$k$-colorings of Cartesian products of S-prime graphs.

**Theorem 2** (Path-$k$-coloring of Cartesian products of S-prime Graphs)**.** *Let $G = \Box_{j=1}^n S_j$ be a Cartesian product of S-prime graphs, and let $F$ be a $k$-coloring of $G$. Then $F$ is a path-$k$-coloring of $G$ if and only if there exists an index set $I \subseteq I_n$ such that the following two conditions hold for the graph $H$ defined as $H = \Box_{i \in I} S_i$ for $I \neq \emptyset$ and $H = K_1$ for $I = \emptyset$.*

1. *$F(a) = F(b)$ for all $a, b \in V(H^x)$ for all $x \in V(G)$ and*
2. *$F(a) \neq F(b)$ for all $a \in V(H^x)$ and $b \in V(H^y)$ with $H^x \neq H^y$.*

*The coloring $F$ consists of $k = |V(G)|/|V(H)|$ distinct colors. $F$ is nontrivial if and only if $I \neq I_n$ and $I \neq \emptyset$.*

*Proof.* Let $F$ be an arbitrary path-$k$-coloring of $G$. If $F$ is trivial, then it follows that $k = 1$ or $k = |V(G)|$ and thus we can conclude that $I = I_n$ or $I = \emptyset$, respectively. In both cases, conditions (1) and (2) are satisfied. If $F$ is nontrivial, then $k \leq |V(G)| - 1$ and there are two vertices with the same color. Conditions (1) and (2) now follow directly from Lemma 6 and Lemma 7.

We will prove the converse by contraposition. Thus assume that $F$ satisfied properties (1) and (2) for some $I \subseteq I_n$ and $F$ is not a path-$k$-coloring of $G$. Thus, there must be a well colored path $P_{u,v}$ between two vertices $u$ and $v$ with $F(u) = F(v)$. If there is an edge $(a,b) \in P_{u,v}$ such that $(a,b)$ is contained in an $H$-layer $H^x$ for some $x \in V(G)$ we would contradict Condition (1). Thus assume there is no edge $(a,b) \in P_{u,v}$ that lies in any $H$-layer. Notice that this implies that $u$ and $v$ are not contained in the same $H$-layer, otherwise some edge $(a,b) \in P_{u,v}$ must be an edge of an $H$-layer, by definition of the Cartesian product. Since $P_{u,v}$ is a well colored path between $u$ and $v$ with $F(u) = F(v)$ and $H^u \neq H^v$, we contradict Condition (2).

It remains to show that $F$ consists of $k = |V(G)|/|V(H)|$ different colors. For $I = I_n$ and $I = \emptyset$ this assertion is trivially true. Therefore assume $I \neq I_n$ and $I \neq \emptyset$. Condition (2) implies that all pairwise different $H$-layer are colored differently and from Condition (1) we can conclude that all vertices in between each $H$-layer have the same color. Thus we have just as many colors as $H$-layers exists. In a Cartesian product $G = H \Box H'$ the number of different $H$-layers is $|V(H')| = |V(G)|/|V(H)|$ and thus $k = |V(G)|/|V(H)|$.

Finally, we have to show that $F$ is nontrivial if and only if $I \neq I_n$ and $I \neq \emptyset$. If $F$ is nontrivial the assumption is already shown at the beginning of this proof. Thus assume now that $I = I_n$, i.e., $H = \Box_{i \in I} S_i = G$. Condition (1) implies that all vertices $v \in V(G)$ have the same color and hence $k = 1$, contradicting that $F$ is nontrivial. Now let $I = \emptyset$, i.e. $H = K_1$. As for all vertices $v, x \in V(G)$ holds $v \in V(K_1^x)$ if and only if $v = x$, we can conclude that $F(a) \neq F(b)$ for all $a, b \in V(G)$. Hence $k = |V(G)|$, again contradicting that $F$ is nontrivial. $\Box$

In the following, let $F_I$ denote a path-$k$-coloring $F$ of a Cartesian product $G$ of S-prime graphs $S_i$ that satisfies the conditions of Theorem 2 with index set $I$. We can now proceed proving the main result of this contribution.

**Theorem 3.** *The diagonalized Cartesian Product of S-prime graphs is S-prime.*

*Proof.* Let $G = H \cup (uv)$ be a diagonalized Cartesian product of graphs $S_i$, i.e., $H = \Box_{i=1}^n S_i$ is a Cartesian product of S-prime graphs and the vertices $u$ and $v$ have maximal distance in $H$. Lemma 1 shows that any nontrivial path-$k$-coloring of $G$ gives rise to a nontrivial path-$k$-coloring of $H$, which in turn implies that there is a nontrivial subset $I \subset I_n$ and an according nontrivial path-$k$-coloring $F_I$ such that the conditions of Theorem 2 are satisfied for $H$. We can conclude that $F_I(u) \neq F_I(v)$, since otherwise the coloring of $H$ is trivial with $k = 1$ according to Corollary 3 and $F_I$ would be constant. Let $H_I$ denote the Cartesian product $\Box_{i \in I} S_i$ of prime factors of $G$ and let $H_I^u$ and $H_I^v$ be the $H_I - layer$ containing $u$ and $v$, respectively. Clearly, $H_I^u \neq H_I^v$, since $I \neq \{1, \ldots, n\}$, by definition of the Cartesian product and since $u$ and $v$ have maximal distance in $H$. Let $\tilde{u} \in V(S_i^v)$ be the vertex with coordinates $c_i(\tilde{u}) = c_i(u)$ for all $i \in I$. Note that $v \neq \tilde{u}$, because $c_i(\tilde{u}) = c_i(u) \neq c_i(v)$ for all $i \in I$, otherwise $u$ and $v$ would not have maximal distance.

Let $P_{u,\tilde{u}}$ be a path between $u$ and $\tilde{u}$ such that for all vertices $x \in P_{u,\tilde{u}}$ holds $c_i(x) = c_i(u)$ for all $i \in I$. Thus no edge of any $H_I$-layer is contained in this path $P_{u,\tilde{u}}$. From Theorem 2 and the fact that $F_I$ is nontrivial, it follows that $F_I(a) \neq F_I(b)$ for all $a \in V(H_I^x)$ and $b \in V(H_I^y)$ with $H_I^x \neq H_I^y$. This is true in particular also for any two distinct vertices $a$ and $b$ in the path $P_{u,\tilde{u}}$, since $H_I^a \neq H_I^b$ by choice of the coordinates. Thus $P_{u,\tilde{u}}$ is well colored. Moreover it holds $F_I(u) \neq F_I(\tilde{u})$.

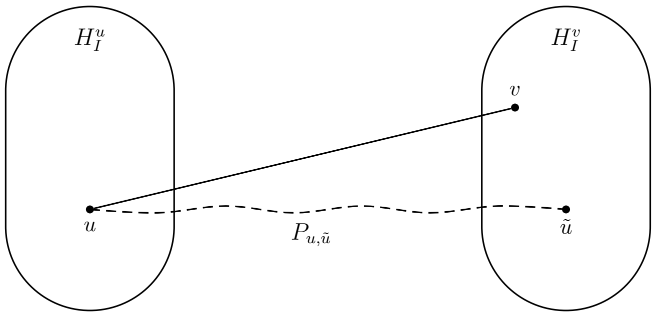


Figure 6: Sketch of the proof of Theorem 3. The $H_I$-layers $H_I^u$ and $H_I^v$ are connected by a well-colored path $P_{u,\tilde{u}}$ with distinct colors at the endpoints, $F_I(u) \neq F_I(\tilde{u})$. The path $P^* = P_{u,\tilde{u}} \cup (u,v)$ is well colored, but $F_I(u) = F_I(v)$, i.e., $F_I$ is not a path-$k$-coloring.

Now consider the path $P^* = P_{u,\tilde{u}} \cup (u,v)$ in $G$, which is by construction a well colored path from $v$ to $\tilde{u}$. However, $F_I(v) = F_I(\tilde{u})$. Thus $F_I$ is not a path-$k$-coloring of $G$ for any nontrivial $I \subset I_n$. Theorem 1 and Lemma 1 imply that $G = H \cup (uv)$ is S-prime, from what the assumption follows. □

**Corollary 4.** *Diagonalized Hamming graphs, and thus diagonalized Hypercubes, are S-prime.*

We conclude our presentation with an example that shows that not every diagonalized Cartesian product is S-prime, see Figure 7, and open problems:

**Problem 1.** *Are there other classes of diagonalized Cartesian products that are S-prime?*

**Problem 2.** *Which of the (known) families of S-prime graphs that are not diagonalized Cartesian products can be non-trivially isometrically embedded into diagonalized Cartesian products of S-prime graphs, i.e., they are not contained in single layers?*

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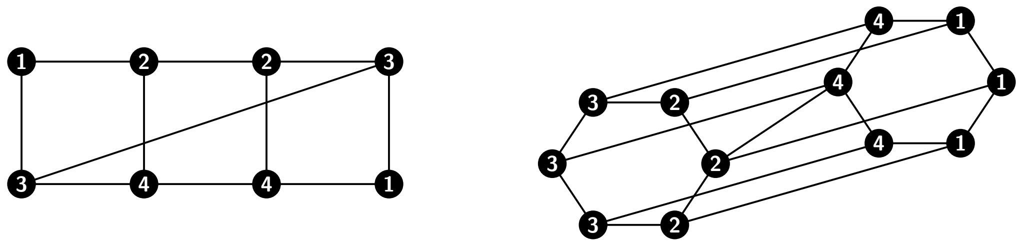

Figure 7: Shown are two diagonalized Cartesian products that have a nontrivial path-4-coloring. Therefore these graphs are S-composite.


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