

Symmetries of the honeycomb toroidal graphs

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Abstract

Honeycomb toroidal graphs are a family of cubic graphs determined by a set of three parameters that have been studied over the last three decades both by mathematicians and computer scientists. They all can be embedded on a torus and coincide with the cubic Cayley graphs of generalized dihedral groups with respect to a set of three reflections. In a recent survey paper B. Alspach gathered most known results on this intriguing family of graphs and suggested a number of research problems regarding them. In this paper we solve two of these problems by determining the full automorphism group of each honeycomb toroidal graph.

KEYWORDS

automorphism, Cayley, cubic, honeycomb toroidal graph

1 | INTRODUCTION

In this short paper we focus on a certain family of cubic graphs with many interesting properties. They are called *honeycomb toroidal graphs*, mainly because they can be embedded on the torus in such a way that the corresponding faces are hexagons. The usual definition of these graphs is purely combinatorial where, somewhat vaguely, the honeycomb toroidal graph $\text{HTG}(m, n, \ell)$ is defined as the graph of order mn having m disjoint “vertical” n -cycles (with n even) such that two consecutive n -cycles are linked together by $n/2$ “horizontal” edges, linking every other vertex of the first cycle to every other vertex of the second one, and where the last “vertical” cycle is linked back to the first one according to the parameter ℓ (see Section 3 for a precise definition). As was shown in [2] these graphs can alternatively be described as Cayley graphs of generalized dihedral groups with respect to a set of three reflections which for instance implies that these graphs are vertex-transitive (see Section 2 for the definition of some terms not defined in Section 1).

It is thus not surprising that this family of graphs has been studied in various papers, both by mathematicians and by computer scientists. The mathematicians are of course always interested in graphs having nice structural properties and a high degree of symmetry. But one of

the more important reasons why these graphs are of particular interest to them is that they stand as the last obstacle to a proof that each bipartite Cayley graph of a generalized dihedral group has the property that any two vertices in different partition sets are linked by a Hamilton path of this graph (see [1,3]). In view of the fact that it is not even known whether all Cayley graphs of dihedral groups possess a Hamilton cycle, this would be a very remarkable result. The main point of interest for computer scientists in these graphs is that small valency and high degree of symmetry make them desirable models for (computer) networks.

In a recent survey paper [1] Alspach discusses the above-mentioned different viewpoints regarding these graphs, gives an indication of how they came about and why they are interesting to researchers from different fields of science, and gathers most of the known results on the topic. Doing so he gives a number of interesting open problems, two of which concern symmetries of these graphs. In [1, Research Problem 4] he suggests the problem of determining the full automorphism group of each $\text{HTG}(m, n, \ell)$. Should one be able to solve this problem, the answer to [1, Research Problem 5], which asks for the classification of all examples with the smallest possible automorphism group (in this case the group acts regularly on the vertex-set of the graph), would of course also be obtained.

The purpose of this paper is to solve these two problems by proving the following two theorems (see Section 5 for the definition of the graphs $\text{GPr}(n)$).

Theorem 1.1. *Let m and n be positive integers, where $n \geq 4$ is even, let $0 \leq \ell \leq n/2$ be an integer of the same parity as m , let $\Gamma = \text{HTG}(m, n, \ell)$ and let the group G be as in (1) of Proposition 3.2. Then Γ is not a normal Cayley graph of G if and only if one of the following holds:*

- $\Gamma = \text{HTG}(1, 6, 3)$ and is isomorphic to the 3-arc-regular complete bipartite graph $K_{3,3}$;
- $\Gamma \in \{\text{HTG}(2, 4, 0), \text{HTG}(1, 8, 3)\}$ and is isomorphic to the 2-arc-regular cube graph;
- $\Gamma = \text{HTG}(1, 14, 5)$ and is isomorphic to the 4-arc-regular Heawood graph;
- $\Gamma \in \{\text{HTG}(1, 16, 5), \text{HTG}(2, 8, 4)\}$ and is isomorphic to the 2-arc-regular Möbius–Kantor graph;
- $\Gamma = \text{HTG}(3, 6, 3)$ and is isomorphic to the 3-arc-regular Pappus graph;
- $mn = 4n'$ for some integer $n' > 2$, either $n = 4$ or $\Gamma \in \{\text{HTG}(1, 4n', 2n' - 1), \text{HTG}(2, 2n', 2)\}$, and Γ is isomorphic to the generalized prism graph $\text{GPr}(n')$, is not arc-transitive and has vertex stabilizers of order $2^{n'-1}$.

Theorem 1.2. *Let m and n be positive integers, where $n \geq 4$ is even, let $0 \leq \ell \leq n/2$ be an integer of the same parity as m and let $\Gamma = \text{HTG}(m, n, \ell)$. If Γ is none of the graphs from Theorem 1.1, then its automorphism group can be determined via the four conditions,*

- (c1) $\gcd(n, \ell + m) = 2m$ and $2mn(\ell^2 + 2m\ell - 3m^2)$,
- (c2) $\gcd(n, \ell - m) = 2m$ and $2mn(\ell^2 - 2m\ell - 3m^2)$,
- (c3) $\ell \in \{0, n/2\}$,
- (c4) $\gcd(n, \ell + m) = 2m = \gcd(n, \ell - m)$ and $2mn(\ell^2 + 3m^2)$,

where

- Γ is 2-arc-regular if and only if any two (and thus all) of (c1), (c2), (c3), and (c4) hold, which occurs if and only if Γ is one of $\text{HTG}(m, 2m, m)$ with $m \geq 2$, $m \neq 3$, and $\text{HTG}(m, 6m, 3m)$ with $m \geq 2$;

- Γ is 1-arc-regular if and only if (c4) holds, but none of (c1), (c2), and (c3) holds;
- Γ is not arc-transitive with vertex stabilizers of order 2 if and only if precisely one of (c1), (c2), and (c3) holds;
- $\text{Aut}(\Gamma)$ is regular on Γ if and only if none of (c1), (c2), (c3), and (c4) holds.

We remark that while each of $\text{HTG}(2, 4, 0)$, $\text{HTG}(1, 8, 3)$, and $\text{HTG}(2, 4, 2)$ is isomorphic to the cube graph, the presentation $\text{HTG}(2, 4, 2)$ results in this graph being a normal Cayley graph of the corresponding group G from (1) (the elementary abelian group of order 8), while the remaining two presentations do not (here the group G is the dihedral group of order 8).

2 | PRELIMINARIES

Throughout the paper all graphs are assumed to be finite, connected, and undirected. Adjacency is denoted by \sim and the edges are usually given as unordered pairs of vertices.

For an integer n the ring of residue classes modulo n is denoted by \mathbb{Z}_n . Therefore, all computations involving elements from \mathbb{Z}_n are performed modulo n .

For an abelian group A the *generalized dihedral group* corresponding to A is the group of order $2|A|$ generated by A and an involution t not in A such that $tat = a^{-1}$ for all $a \in A$.

For a group G and an inverse-closed subset $S \subset G \setminus \{1\}$ the *Cayley graph* $\text{Cay}(G; S)$ of G with respect to S has vertex-set G and edge-set $\{\{g, gs\} : g \in G, s \in S\}$. The graph $\text{Cay}(G; S)$ is a *normal* Cayley graph of G if the left regular representation G_L of G is a normal subgroup of the automorphism group $\text{Aut}(\text{Cay}(G; S))$.

A graph Γ is *vertex-transitive* if the automorphism group $\text{Aut}(\Gamma)$ of Γ acts transitively on the vertex-set of Γ . For $s \geq 1$ an *s-arc* of Γ is a sequence of $s + 1$ vertices such that any consecutive two are adjacent and any consecutive three are pairwise distinct. A graph Γ is *s-arc-transitive* if $\text{Aut}(\Gamma)$ acts transitively on the set of all *s-arcs* of Γ . If this action is regular, Γ is said to be *s-arc-regular*. The term 1-arc-transitive is abbreviated to *arc-transitive*.

3 | THE HTG GRAPHS

We now review the definition of the honeycomb toroidal graphs and their presentation as Cayley graphs of an appropriate generalized dihedral group. We also fix some terminology pertaining to the two viewpoints that we will be using throughout the rest of the paper. The following is simply a restatement of the definition given in [2].

Construction 3.1. *Let m and n be positive integers, where $n \geq 4$ is even. For each integer ℓ with $0 \leq \ell \leq n - 1$, where ℓ is of the same parity as m , the honeycomb toroidal graph $\text{HTG}(m, n, \ell)$ is the cubic graph with vertex-set $\{\langle i, j \rangle : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and the following adjacencies:*

- $\langle i, j \rangle \sim \langle i, j \pm 1 \rangle$ for all $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$;
- $\langle i, j \rangle \sim \langle i + 1, j \rangle$ for all $i \in \mathbb{Z}_m \setminus \{m - 1\}, j \in \mathbb{Z}_n$ with i and j of different parity;
- $\langle m - 1, j \rangle \sim \langle 0, j + \ell \rangle$ for all $j \in \mathbb{Z}_m$ of the same parity as m .

The reader will notice that $\text{HTG}(m, n, \ell) \cong \text{HTG}(m, n, n - \ell)$, and so we lose nothing by assuming $\ell \leq n/2$, which is what we will usually do (in this case the HTG graph is said to be in *normal form* [1]).

In [2] it was shown that each $\text{HTG}(m, n, \ell)$ is isomorphic to a Cayley graph of a generalized dihedral group. In particular, the following result was proved.

Proposition 3.2 (Alspach and Dean [2, Theorem 3.4]). *Let m and n be positive integers, where $n \geq 4$ is even, and let $0 \leq \ell \leq n - 1$ be an integer of the same parity as m . Let*

$$G = \langle t, x, y \mid t^2 = x^{n/2} = 1, y^m = x^{(\ell+m)/2}, txt = x^{-1}, tyt = y^{-1}, xy = yx \rangle. \tag{1}$$

Then the honeycomb toroidal graph $\text{HTG}(m, n, \ell)$ is isomorphic to the Cayley graph $\text{Cay}(G; \{t, tx, ty\})$.

Observe that the group G is a generalized dihedral group, where $\langle x, y \rangle$ is the index 2 abelian subgroup (of order $mn/2$). It is easy to verify that the orders of x, y , and $x^{-1}y$ are

$$|x| = n/2, \quad |y| = \frac{mn}{\gcd(n, \ell + m)}, \quad \text{and} \quad |x^{-1}y| = \frac{mn}{\gcd(n, \ell - m)}. \tag{2}$$

The following detail from the proof of [2, Theorem 3.4] will be important for us. One of the isomorphisms from $\text{HTG}(m, n, \ell)$ to $\text{Cay}(G; \{t, tx, ty\})$ is given by the following correspondence between the vertices of these two graphs. For each $i \in \{0, 1, \dots, m - 1\}$ let $i_n \in \{0, 1, \dots, n - 1\}$ be such that $i \equiv i_n \pmod{n}$ (in other words, i_n is i computed modulo n). Then for each $i \in \{0, 1, \dots, m - 1\}$ the vertex $\langle i, i_n \rangle$ corresponds to y^i and for each $j \in \{0, 1, \dots, n/2 - 1\}$ the vertex $\langle i, (i + 2j)_n \rangle$ corresponds to $x^j y^i$, while the vertex $\langle i, (i + 2j + 1)_n \rangle$ corresponds to $x^j y^i t = tx^{-j} y^{-i}$.

Throughout the paper we will constantly be switching between these two viewpoints of the HTG graphs. More precisely, whenever we have an HTG graph Γ , we think of its vertices as being the pairs $\langle i, j \rangle$ from Construction 3.1 and at the same time as being the elements of the above group G via the described correspondence. With this in mind we let

$$\mathcal{R} = \{\{g, gt\} : g \in G\}, \quad \mathcal{B} = \{\{g, gtx\} : g \in G\}, \quad \text{and} \quad \mathcal{G} = \{\{g, gty\} : g \in G\}, \tag{3}$$

and call the members of \mathcal{R}, \mathcal{B} , and \mathcal{G} the *red, blue, and green* edges, respectively (see Figure 1 for two examples of how the edges are colored). In this respect we are thus viewing $\text{HTG}(m, n, \ell)$ as what is known in the literature as a Cayley colored graph. Note that each vertex of Γ is incident to one edge of each of the three colors. Moreover, for any pair of colors the subgraph consisting of all the edges of these two colors is a disjoint union of cycles (all of which have the same even length half of, which equals one of the numbers from Equation 2, depending on the two chosen colors) which in the whole graph are linked together by the edges of the third color in a cyclic fashion.

Let us also point out that, since the index 2 abelian subgroup $\langle x, y \rangle$ of G of course acts semiregularly with two orbits on the vertex-set of Γ , the HTG graphs are also so-called bi-Cayley graphs on abelian groups (see, e.g., [4,15,16]). Moreover, as the two orbits of $\langle x, y \rangle$ are independent sets, they are also so-called Haar graphs (see, e.g., [6,7]) of abelian groups. One could thus make use of some known results, in particular those of [4] where symmetries of

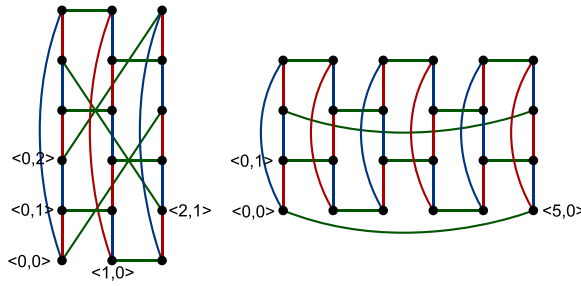


FIGURE 1 The honeycomb toroidal graphs HTG(3, 6, 3) and HTG(6, 4, 0) [Color figure can be viewed at wileyonlinelibrary.com]

edge-transitive bi-Cayley graphs were studied extensively. Nevertheless, as we will see, in this special setting a direct approach works just fine.

4 | THE COLOR-PERMUTING AUTOMORPHISMS

In this section we investigate the automorphisms of the HTG graphs that permute the colors of the edges as given in (3). More precisely, let $\Gamma = \text{HTG}(m, n, \ell)$ and let \mathcal{R}, \mathcal{B} , and \mathcal{G} be as in (3). We investigate the automorphisms of Γ which preserve the set $\{\mathcal{R}, \mathcal{B}, \mathcal{G}\}$, that is, the automorphisms of Γ for which whenever two edges are of the same color, the images of these two edges are also of the same color. Such automorphisms are said to be *color-permuting* in [8,9] where this concept was studied in some detail.

Let G be as in (1) and let G_L be the left regular representation of G . Of course, each element of G_L is color-permuting (in fact, \mathcal{R}, \mathcal{B} , and \mathcal{G} are the three orbits of G_L on the edge-set of Γ). Denote the set of all automorphisms of G which fix the set $\{t, tx, ty\}$ setwise by $\text{Aut}(G; \{t, tx, ty\})$. Then each element of $\text{Aut}(G; \{t, tx, ty\})$ is a color-permuting automorphism of Γ which fixes the vertex 1. As in our case each element of the connection set $\{t, tx, ty\}$ is an involution, it is easy to verify that the converse also holds (but see also [8, Lemma 2.1] or [9, Section 5]). The set of all color-permuting automorphisms of Γ that fix the vertex 1 thus coincides with $\text{Aut}(G; \{t, tx, ty\})$. It now easily follows that the group $\text{Aut}_c(\Gamma)$ of all color-permuting automorphisms of Γ is the semidirect product $G_L \rtimes \text{Aut}(G; \{t, tx, ty\})$. In the words of [9], all elements of $\text{Aut}_c(\Gamma)$ are of affine type. It is well known and easy to see (but see, e.g., the seminal paper by Xu [14] in which the notion of a normal Cayley graph was introduced) that for a Cayley graph $\Gamma' = \text{Cay}(H; S)$ the group H_L is normal in $\text{Aut}(\Gamma')$ if and only if $\text{Aut}(\Gamma') = H_L \rtimes \text{Aut}(H; S)$. Therefore, in our setting the graph $\Gamma = \text{HTG}(m, n, \ell)$ is a normal Cayley graph of G from (1) if and only if $\text{Aut}(\Gamma) = \text{Aut}_c(\Gamma)$.

In the rest of this section we determine the group $\text{Aut}(G; \{t, tx, ty\})$ (and consequently also $\text{Aut}_c(\Gamma)$). Of course, as the connection set consists of three involutions we have that $\text{Aut}(G; \{t, tx, ty\})$ is isomorphic to a subgroup of the symmetric group S_3 . Note that since the set $\{t, tx, ty\}$ generates G an automorphism of G is completely determined by its action on t, tx , and ty . Therefore, a permutation φ of the set $\{t, tx, ty\}$ extends to an automorphism of G if and only if the elements $\varphi(t), \varphi(x) := \varphi(t)\varphi(tx)$ and $\varphi(y) := \varphi(t)\varphi(ty)$ satisfy the defining relations of G from (1). Now, the product of any two elements of $\{t, tx, ty\}$ is contained in the index 2 abelian subgroup $\langle x, y \rangle$ of G and taking any $a, b \in \langle x, y \rangle$ we have that $(ta)b(ta) = b^{-1}$. This shows that the only nontrivial conditions for φ to extend to an automorphism of G are $|\varphi(x)| = n/2$ and $(\varphi(y))^m = (\varphi(x))^{(\ell+m)/2}$. Together with (2) this proves the following.

Lemma 4.1. *Let the group G be as in (1). Then a permutation φ of the set $\{t, tx, ty\}$ extends to an automorphism of G if and only if*

$$|\varphi(t)\varphi(tx)| = n/2, \quad |\varphi(t)\varphi(ty)| = \frac{mn}{\gcd(n, \ell + m)}, \quad |\varphi(tx)\varphi(ty)| = \frac{mn}{\gcd(n, \ell - m)} \quad (4)$$

and

$$(\varphi(t)\varphi(ty))^m = (\varphi(t)\varphi(tx))^{(\ell+m)/2}. \quad (5)$$

We first consider each of the three possibilities of fixing one of the elements of $\{t, tx, ty\}$ and interchanging the other two, and then finally also consider the possibility of permuting all three generators $t, tx,$ and ty in a cycle of length 3.

Lemma 4.2. *Let the group G be as in (1). Then there exists an automorphism in $\text{Aut}(G; \{t, tx, ty\})$ fixing t and interchanging tx with ty if and only if*

$$\gcd(n, \ell + m) = 2m \quad \text{and} \quad 2mnl(\ell^2 + 2m\ell - 3m^2).$$

Proof. In this case condition (4) is equivalent to $n/2 = mn/\gcd(n, \ell + m)$, that is, $\gcd(n, \ell + m) = 2m$, while (5) reads $x^m = y^{(\ell+m)/2}$. Assuming the first of these holds we have that $2m$ divides n and $\ell + m = 2qm$ for some q coprime to $n/(2m)$. Then (1) implies that (5) is in fact

$$x^m = y^{(\ell+m)/2} = y^{qm} = (y^m)^q = x^{q(\ell+m)/2},$$

and so (5) reads $x^m = x^{(\ell+m)^2/(4m)}$, which holds if and only if $n/2$ divides $(\ell^2 + 2m\ell - 3m^2)/4m$. \square

Lemma 4.3. *Let the group G be as in (1). Then there exists an automorphism in $\text{Aut}(G; \{t, tx, ty\})$ fixing tx and interchanging t with ty if and only if*

$$\gcd(n, \ell - m) = 2m \quad \text{and} \quad 2mnl(\ell^2 - 2m\ell - 3m^2).$$

Proof. In this case (4) is equivalent to $\gcd(n, \ell - m) = 2m$, while (5) reads $y^{-m} = (y^{-1}x)^{(\ell+m)/2}$, which is equivalent to $x^{(\ell+m)/2} = y^{(\ell-m)/2}$. If $\gcd(n, \ell - m) = 2m$, then $2m|n$ and $\ell - m = 2qm$ for some q coprime to $n/(2m)$. Then $(\ell + m)/2 = (q + 1)m$, and so (1) implies that (5) is equivalent to

$$x^{(q+1)m} = y^{qm} = x^{q(\ell+m)/2} = x^{q(q+1)m},$$

which holds if and only if $1 = x^{(q+1)(q-1)m} = x^{(\ell^2 - 2m\ell - 3m^2)/(4m)}$. The result now follows. \square

Lemma 4.4. *Let the group G be as in (1). Then there exists an automorphism in $\text{Aut}(G; \{t, tx, ty\})$ fixing ty and interchanging t with tx if and only if $\ell \in \{0, n/2\}$.*

Proof. In this case (4) is equivalent to $\gcd(n, \ell + m) = \gcd(n, \ell - m)$, while (5) reads $(x^{-1}y)^m = x^{-(\ell+m)/2}$. By (1) the latter is equivalent to $x^{(\ell+m)/2} = y^m = x^{(m-\ell)/2}$, that is, $x^\ell = 1$. Since this holds if and only if $\ell \in \{0, n/2\}$, we only need to verify that $\gcd(n, n/2 + m) = \gcd(n, n/2 - m)$ holds. As $(n/2 + m) + (n/2 - m) = n$, this is clear. \square

Lemma 4.5. *Let the group G be as in (1). Then there exists an automorphism in $\text{Aut}(G; \{t, tx, ty\})$ of order 3 if and only if*

$$\gcd(n, \ell + m) = \gcd(n, \ell - m) = 2m \quad \text{and} \quad 2mn | (\ell^2 + 3m^2).$$

Proof. Observe that such an automorphism exists if and only if there is an automorphism in $\text{Aut}(G; \{t, tx, ty\})$ mapping t to tx , tx to ty and ty back to t . In this case (4) is equivalent to $\gcd(n, \ell + m) = \gcd(n, \ell - m) = 2m$. This time (5) reads $x^{-m} = (x^{-1}y)^{(\ell+m)/2}$. Assuming (4) holds we have that $2m$ divides n and $\ell = (2q + 1)m$ for some q coprime to $n/(2m)$. Then (1) implies that (5) is equivalent to

$$x^{qm} = x^{(\ell-m)/2} = y^{(\ell+m)/2} = y^{(q+1)m} = x^{(q+1)^2m},$$

which holds if and only if $1 = x^{(q^2+q+1)m} = x^{(\ell^2+3m^2)/(4m)}$. \square

Corollary 4.6. *Let the group G be as in (1). Then $\text{Aut}(G; \{t, tx, ty\}) \cong S_3$ if and only if $n = 2\ell$ and either $\ell = m$ or $\ell = 3m$.*

Proof. Suppose $\text{Aut}(G; \{t, tx, ty\}) \cong S_3$. Then Lemma 4.4 implies that $\ell \in \{0, n/2\}$ and Lemma 4.5 implies that $\gcd(n, \ell + m) = \gcd(n, \ell - m) = 2m$ and $2mn | (\ell^2 + 3m^2)$. It follows that $\ell \neq 0$, and so $\ell = n/2$. Since $2mn$ divides $4(\ell^2 + 3m^2) = n^2 + 12m^2$ and $2m | n$, it follows that $2mn | 12m^2$, that is, $n | 6m$. Together with $2m | n$ this yields $n \in \{2m, 6m\}$, and so $\ell \in \{m, 3m\}$.

For the converse one simply has to verify that setting $n = 2\ell$ and either $\ell = m$ or $\ell = 3m$ the conditions of Lemmas 4.2 and 4.4 are both satisfied, which is clear. \square

Under the assumption that Theorem 1.1 holds, the results of this section thus prove Theorem 1.2 (recall that $\text{HTG}(m, n, \ell)$ is a normal Cayley graph of G from (1) if and only if $\text{Aut}(G; \{t, tx, ty\}) = \text{Aut}(\Gamma)_1$).

5 | ADDITIONAL AUTOMORPHISMS

In view of the results of Section 4 it remains to classify those $\Gamma = \text{HTG}(m, n, \ell)$ for which $\text{Aut}(\Gamma) \neq \text{Aut}_c(\Gamma)$ and then determine the automorphism groups of these examples. As we shall see there is only a handful of well-known small examples and a very specific infinite family of such graphs. We start with an easy observation (recall the definition of the colors from Equation 3).

Lemma 5.1. *Let m and n be positive integers, where $n \geq 4$ is even, let $0 \leq \ell \leq n - 1$ be an integer of the same parity as m , and let $\Gamma = \text{HTG}(m, n, \ell)$. If $\alpha \in \text{Aut}(\Gamma)$ is such that there is no edge e such that e and $\alpha(e)$ are of the same color, then $\alpha \in \text{Aut}_c(\Gamma)$.*

Proof. The assumption that α preserves the color of no edge of Γ and the fact that the three edges incident to a given vertex have different colors imply that for each vertex v and the three edges incident to it, say e , e' , and e'' , knowing the color of one of $\alpha(e)$, $\alpha(e')$, and $\alpha(e'')$ completely determines the color of all three of these images. As Γ is connected, the result follows easily. \square

It is well known that the graphs $\text{HTG}(m, n, \ell)$ are bipartite (but recall also Proposition 3.2) and are of girth at most 6. In fact, the product of the three elements from $\{t, tx, ty\}$ in any order (where G is as in Equation 1) is an involution, and so starting at any vertex of the graph and then following any walk that has the property that each three consecutive edges on it are of three different colors results in a 6-cycle. We call all such 6-cycles *generic*. The set of all generic 6-cycles is clearly a G_L -orbit and there are precisely two generic 6-cycles through each edge.

Before stating and proving the next easy but useful result we review the definition of the *generalized prism* $\text{GPr}(n)$ from [5], where cubic vertex-transitive graphs of girth at most 5 were characterized. The graph $\text{GPr}(n)$, where $n \geq 2$, has vertex-set $\{(i, j) : i \in \mathbb{Z}_2, j \in \mathbb{Z}_{2n}\}$, each vertex (i, j) is adjacent to $(i, j \pm 1)$ and in addition (i, j) is adjacent to $(i + 1, j + 1)$ for all even j (see Figure 2 where the isomorphic graphs $\text{GPr}(4)$, $\text{HTG}(4, 4, 2)$, $\text{HTG}(1, 16, 7)$, and $\text{HTG}(2, 8, 2)$ are depicted). These graphs might also be called the *split wreath graphs* as $\text{GPr}(n)$ can be obtained from the well-known *wreath graph* $W(n) = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n; \{(0, \pm 1), (1, \pm 1)\})$ by performing the “splitting” construction with respect to the cycle decomposition consisting of all the “natural” 4-cycles of $W(n)$ (see [11, Construction 11] for details).

Proposition 5.2. *Let m and n be positive integers, where $n \geq 4$ is even, let $0 \leq \ell \leq n - 1$ be an integer of the same parity as m , and let $\Gamma = \text{HTG}(m, n, \ell)$. Let \mathcal{R} , \mathcal{B} , and \mathcal{G} be as in (3). Then either each automorphism of Γ fixing setwise at least one of the sets \mathcal{R} , \mathcal{B} , and \mathcal{G} is color-permuting, or mn is divisible by 4 and $\Gamma \cong \text{GPr}(mn/4)$.*

Proof. Suppose Γ is not a generalized prism. We show that in this case each $\alpha \in \text{Aut}(\Gamma)$ fixing the set \mathcal{R} setwise is color-permuting (the possibilities that it fixes setwise \mathcal{B} or \mathcal{G} are dealt with analogously).

To this end we first show that if $\alpha \in \text{Aut}(\Gamma)$ with $\alpha(\mathcal{R}) = \mathcal{R}$ fixes a vertex and all of its neighbors, then $\alpha = 1$. Let α be such an automorphism and let g be a corresponding vertex. Then α fixes the entire *blue–green* cycle C (the edges are alternatingly blue and green) containing g pointwise. It thus fixes each neighbor of gtx as well as each neighbor of gty . If C is of length mn we are done. So suppose this is not the case. Let $2s$ be the length of C and consider the blue–green cycle C' containing gt . Because of the generic 6-cycles there is a set of at least s red edges joining the vertices of C to C' (if $mn = 4s$ there are $2s$ of them). As Γ is not

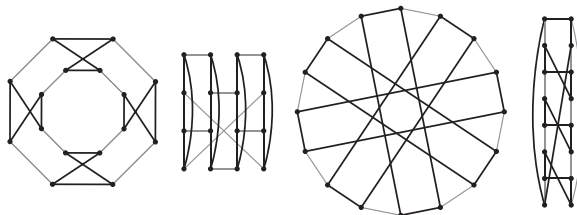


FIGURE 2 The isomorphic graphs $\text{GPr}(4)$, $\text{HTG}(4, 4, 2)$, $\text{HTG}(1, 16, 7)$, and $\text{HTG}(2, 8, 2)$

a generalized prism we have $s \geq 3$, but then the fact that C is fixed pointwise implies that C' must also be fixed pointwise. Therefore, if α fixes g and all of its neighbors, it also fixes the neighbors of each of the three neighbors of g . As Γ is connected, $\alpha = 1$.

To complete the proof let $\alpha \in \text{Aut}(\Gamma)$ with $\alpha(\mathcal{R}) = \mathcal{R}$. If $\alpha(\mathcal{B}) = \mathcal{G}$ there is nothing to prove, so assume there is some $e \in \mathcal{B}$ with $\alpha(e) \in \mathcal{B}$. Let $g, h \in G$ be such that $e = \{g, gtx\}$ and $\alpha(g) = h$. Then $\alpha(gtx) = htx$, and so letting $\beta \in G_L$ correspond to gh^{-1} the automorphism $\beta\alpha$ fixes g and each of its three neighbors. As $\beta\alpha(\mathcal{R}) = \mathcal{R}$, the above argument implies that $\alpha = \beta^{-1} \in \text{Aut}_c(\Gamma)$. □

It thus makes sense to first identify the HTG graphs (of girth 4) which are isomorphic to a generalized prism graph. As was pointed out by Alspach [1] the HTG graphs of girth 4 are easily identified (recall that we lose nothing by assuming $\ell \leq n/2$).

Proposition 5.3 (Alspach [1, Theorem 5.1]). *Let m and n be positive integers, where $n \geq 4$ is even, let $0 \leq \ell \leq n/2$ be an integer of the same parity as m , and let $\Gamma = \text{HTG}(m, n, \ell)$. Then Γ is of girth 4 if and only if one of the following holds:*

- (i) $n = 4$;
- (ii) $m = 1, n \geq 6$, and $\ell = 3$;
- (iii) $m = 1, n \geq 6, n \equiv 2 \pmod{4}$, and $\ell = n/2$;
- (iv) $m = 1, n \geq 8, n \equiv 0 \pmod{4}$, and $\ell = (n - 2)/2$;
- (v) $m = 2, n \geq 6$, and $\ell \in \{0, 2\}$.

It is not difficult to see that for each $m \geq 2$ and $\ell \in \{0, 1, 2\}$, where m and ℓ are of the same parity, $\text{HTG}(m, 4, \ell) \cong \text{GPr}(m)$. Similarly, $\text{HTG}(1, n, (n - 2)/2) \cong \text{GPr}(n/4)$ for each $n \geq 8$ with $n \equiv 0 \pmod{4}$. Finally, $\text{HTG}(2, n, 2) \cong \text{GPr}(n/2)$ for each even $n \geq 6$. We leave the easy verifications of these claims to the reader (but see Figure 2). It is also not difficult to see that none of the remaining graphs from Proposition 5.3 is a generalized prism graph. In fact, for each even $n \geq 6$ the graphs $\text{HTG}(2, n, 0)$ and $\text{HTG}(1, 2n, 3)$ are both isomorphic to the well-known prism graph $\text{Pr}(n) = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n; \{(1, 0), (0, \pm 1)\})$ of order $2n$, while for each odd $n \geq 3$ the graphs $\text{HTG}(1, 2n, n)$ and $\text{HTG}(1, 2n, 3)$ are both isomorphic to the well-known Möbius ladder $\text{Ml}(n) = \text{Cay}(\mathbb{Z}_{2n}; \{\pm 1, n\})$ of order $2n$. With the exception of $\text{Ml}(3) \cong K_{3,3}$ (which clearly admits automorphisms that are not color-permuting) all of these prisms and Möbius ladders have edges of two different types regarding the 4-cycles—one-third of them lie on two 4-cycles each, while the remaining ones lie on one 4-cycle each. This clearly shows that the vertex stabilizers in the automorphism group of each of these graphs Γ are of order 2, and so Lemmas 4.2, 4.3, and 4.4 imply that for these graphs $\text{Aut}(\Gamma) = \text{Aut}_c(\Gamma)$.

For the rest of this section we thus assume that the graphs $\text{HTG}(m, n, \ell)$ we are dealing with are of girth 6. We show that if such a graph admits an automorphism that is not color-permuting, it has a lot of symmetry and contains many 6-cycles.

Proposition 5.4. *Let m and n be positive integers, where $n \geq 6$ is even, and let $0 \leq \ell \leq n/2$ be an integer of the same parity as m such that $\Gamma = \text{HTG}(m, n, \ell)$ is of girth 6. Let \mathcal{R}, \mathcal{B} , and \mathcal{G} be as in (3). If $\text{Aut}(\Gamma) \neq \text{Aut}_c(\Gamma)$ then Γ is 2-arc-transitive and each 3-path of Γ lies on a 6-cycle.*

Proof. Suppose then that there exists some $\alpha \in \text{Aut}(\Gamma) \setminus \text{Aut}_c(\Gamma)$. By Proposition 5.2 none of the sets \mathcal{R}, \mathcal{B} , and \mathcal{G} is preserved by α , and so the fact that they are orbits of the

vertex-transitive group G_L , where G is as in (1), implies that Γ is edge- and thus also arc-transitive. By Lemma 5.1 there is a set in $\{\mathcal{R}, \mathcal{B}, \mathcal{G}\}$, say \mathcal{R} , such that there exist edges $e_1, e_2 \in \mathcal{R}$ with $\alpha(e_1) \in \mathcal{R}$ and $\alpha(e_2) \notin \mathcal{R}$. Because of the red–green and red–blue cycles there is a walk from e_1 to e_2 in Γ such that every other edge of this walk is red (recall that the three edges incident to any given vertex are of distinct colors). This shows that there in fact exists a 3-path P whose initial and terminal edges e_1 and e_2 are both red with $\alpha(e_1) \in \mathcal{R}$ and $\alpha(e_2) \notin \mathcal{R}$. But then $\alpha(P)$ has all three edges of different colors and thus lies on a generic 6-cycle, implying that P also lies on a 6-cycle. There is thus a generic and a nongeneric 6-cycle through the 2-path consisting of e_1 and e , where e is the middle edge of P . To complete the proof we thus only need to show that Γ is indeed 2-arc-transitive.

Since G_L preserves the colors and Γ is arc-transitive, it suffices to verify that for some vertex v the above α preserves the color of one edge incident to v while interchanging the colors of the remaining two. If $\alpha(e)$ has the same color as e , we can choose v to be the common endvertex of e and e_2 , while in the other case we can let v be the common endvertex of e and e_1 . \square

It is now simply a matter of determining all HTG graphs of girth 6 with the property that each of their 3-paths lies on a 6-cycle. Given the special structure of the HTG graphs this is fairly easy to do (e.g., considering a red–blue–red 3-path yields $n = 6$ or $m \leq 2$, and then considering a red–green–red 3-path yields that $m > 2$ only for $\text{HTG}(3, 6, 3)$). However, instead of describing the argument we simply rely on a more general result from the literature. Namely, it follows from [4, Lemma 5.3] or [12, Theorem 1] (but see also [10]) that the graph Γ must be the Heawood graph, the Pappus graph or the Möbius–Kantor graph (it can easily be verified that the Desargues graph is not an HTG graph). We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $\Gamma = \text{HTG}(m, n, \ell)$ is not a normal Cayley graph of G if and only if $\text{Aut}(\Gamma) \neq \text{Aut}_c(\Gamma)$. By the previous paragraph the only possible examples of girth 6 are the Heawood graph, the Pappus graph, and the Möbius–Kantor graph. The automorphism groups of these graphs are well known, while the fact that these are indeed precisely the HTG graphs given in the theorem is easily verified. Moreover, as $\text{Aut}(G; \{t, tx, ty\}) \leq S_3$ the only one of these that could possibly be normal with respect to G is the Möbius–Kantor graph $\text{HTG}(1, 16, 5) \cong \text{HTG}(2, 8, 4)$. However, by Corollary 4.6 this graph is also not a normal Cayley graph of G .

The discussion from the paragraph following Proposition 5.3 shows that the only candidates for the girth 4 HTG graphs that are not normal Cayley graphs of G are the ones given in the theorem. That $K_{3,3}$ and the cube graph are indeed isomorphic to the HTG graphs stated in the theorem is easily verified, while their automorphism groups are well known. Corollary 4.6 shows that the cube graph in the presentations $\text{HTG}(2, 4, 0)$ and $\text{HTG}(1, 8, 3)$ is not a normal Cayley graph of the corresponding group G while it is a normal Cayley graph of the corresponding group G if we take the presentation $\text{HTG}(2, 4, 2)$. Finally, note first that for $m \geq 3$ the edges of $\text{GPr}(m)$ are of two different types. One-third of them lies on no 4-cycle while each of the remaining ones lies on a unique 4-cycle. The ones that lie on no 4-cycle thus clearly constitute an orbit of $\text{Aut}(\text{GPr}(m))$, which shows that $\text{Aut}(\text{GPr}(m))$ is isomorphic to the automorphism group of the corresponding wreath graph $W(m)$ (see also [11]). The latter can be determined easily and can be seen to be generated by an elementary abelian 2-group of rank m and a dihedral group of order $2m$ (but see, e.g., [13]). Since the vertex stabilizers are of order

2^{m-1} and $m \geq 3$, the fact that $\text{Aut}(G; \{t, tx, ty\}) \leq S_3$ implies that this graph is not a normal Cayley graph of G . \square

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REFERENCES

1. B. Alspach, *Honeycomb toroidal graphs*, Bull. Inst. Combin. Appl. **91** (2021), 94–114. to appear.
2. B. Alspach and M. Dean, *Honeycomb toroidal graphs are Cayley graphs*, Inform. Process. Lett. **109** (2009), 705–708.
3. B. Alspach, C. C. Chen, and M. Dean, *Hamilton paths in Cayley graphs on generalized dihedral groups*, Ars Math. Contemp. **3** (2010), 29–47.
4. M. Conder, J. X. Zhou, Y. Q. Feng, and M. M. Zhang, *Edge-transitive bi-Cayley graphs*, J. Combin. Theory Ser. B **45** (2020), 264–306.
5. E. Eiben, R. Jajcay, and P. Šparl, *Symmetry properties of generalized graph truncations*, J. Combin. Theory Ser. B **137** (2019), 291–315.
6. I. Estélyi and T. Pisanski, *Which Haar graphs are Cayley graphs?* Electron. J. Combin. **23** (2016), no. 3, Paper 3.10.
7. Y. Q. Feng, I. Kovács, J. Wang, and D. W. Yang, *Existence of non-Cayley Haar graphs*, European J. Combin. **89** (2020), 103146. 12pp.
8. M. L. Fiol, M. A. Fiol, and J. L. A. Yebra, *When the arc-colored line digraph of a Cayley colored digraph is again a Cayley colored digraph*, Ars Combin. **34** (1992), 65–73.
9. A. Hujdurović, K. Kutnar, D. WitteMorris, and J. Morris, *On color-preserving automorphisms of Cayley graphs*, Ars Math. Contemp. **11** (2016), 189–213.
10. K. Kutnar and D. Marušič, *A complete classification of cubic symmetric graphs of girth 6*, J. Combin. Theory Ser. B **99** (2009), 162–164.
11. P. Potočnik, P. Spiga, and G. Verret, *Cubic vertex-transitive graphs on up to 1280 vertices*, J. Symbolic Comput. **50** (2013), 465–477.
12. P. Potočnik and J. Vidali, *Cubic vertex-transitive graphs of girth six*, Preprint, arXiv:2005.01635.
13. C. E. Praeger and M. Y. Xu, *A characterization of a class of symmetric graphs of twice prime valency*, European J. Combin. **10** (1989), 91–102.
14. M. Y. Xu, *Automorphism groups and isomorphisms of Cayley digraphs*, Discrete Math. **182** (1998), 309–319.
15. J. X. Zhou and Y. Q. Feng, *Cubic bi-Cayley graphs over abelian groups*, European J. Combin. **36** (2014), 679–693.
16. J. X. Zhou and Y. Q. Feng, *The automorphisms of bi-Cayley graphs*, J. Combin. Theory Ser. B **116** (2016), 504–532.

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